

250 LECTURES ON MATHEMATICS · PUBLISHED SERIALLY · THREE TIMES EACH MONTH

PRACTICAL MATHEMATICS

ISSUE
No. 8

THEORY AND PRACTICE WITH MILITARY
AND INDUSTRIAL APPLICATIONS

THE CALCULUS

Differential and Integral Calculus

Variables and Limits

Derivatives

Partial Differentiation

Fractional Exponents

Complex Numbers

— ALSO —

Mathematical Tables and Formulas

Self-Tests and Calculus Problems

NORBERT WIENER, Ph.D.
Massachusetts Institute of Technology



35¢

EDITOR: REGINALD STEVENS KIMBALL ED.D.

ISSUE

Practical Mathematics

8

VOLUME

2

REGINALD STEVENS KIMBALL, Editor

	PAGE
Chats with the Editor	i
About Our Authors	iv

TABLE OF CONTENTS

THE CALCULUS

AN INTRODUCTION TO THE CALCULUS	<i>Herbert Harvey</i> 449
ELEMENTS OF THE CALCULUS	<i>Norbert Wiener, Ph.D.</i>
Variables and Limits	459
Derivatives	464
Higher Derivatives	476
Partial Differentiation	485
Integral Calculus	487

FRACTIONAL EXPONENTS	498
--------------------------------	-----

FURTHER STEPS WITH COMPLEX NUMBERS	500
--	-----

THE MEASURING ROD — VIII

APPLICATIONS OF THE CALCULUS TO PRACTICAL PROBLEMS	503
--	-----

SOLUTIONS TO EXERCISES IN ISSUE VII	506
---	-----

ODD PROBLEMS FOR OFF HOURS — VII

A FEW MORE "TEASERS"	509
--------------------------------	-----

STRANGE WAYS WITH NUMBERS — VII

SOME INTERESTING CURVES	510
-----------------------------------	-----

TABLES AND FORMULAS — VIII	512
--------------------------------------	-----

PRACTICAL MATHEMATICS is published three times a month by the National Educational Alliance, Inc., Office of publication at Washington and South Avenues, Dunellen, N. J. Executive and editorial offices, 37 W. 47th St., New York, N. Y. John J. Crawley, President; A. R. Mahony, Vice President and Business Manager; Frank P. Crawley, Treasurer. Issue No. 8, June 20, 1943. Entered as second-class matter April 10, 1943, at the Post Office at Dunellen, N. J., under the Act of March 3, 1879. Printed in the U.S.A. Price in the U.S.A. 35c a copy; annual subscription at the rate of 35c a copy. Contents copyright 1943 by National Educational Alliance, Inc.

CHATS WITH THE EDITOR

BACK in the days when primitive man did much of his counting on his fingers, mathematics was a simple process. Down through the ages, as man advanced upward in his road toward knowledge, new developments were made possible. We who study mathematics in the present age are aghast at the seemingly limitless possibilities which are presented to us by the multiplicity of formulas upon which we may draw. Perhaps it will come as a surprise to some of you to learn that most of mathematical terminology is of comparatively recent development.

Even the ancient Greeks and the Romans, who advanced to a relatively high state of civilization, were ignorant of many of the devices which we now employ to arrive at a solution of our problems. They were ingenious in their efforts to simplify procedures, however.

When a Roman wanted to cast up his accounts, he disdained to count on his fingers, but he did something which to us seems almost as "lazy". He would take a pile of pebbles, adding or subtracting pebbles to indicate changes in his sum. When the heap of pebbles grew too large for easy counting, he would substitute a single pebble to represent a number of others (much as we use tens, hundreds, thousands, to help us in our advance toward larger numbers). His word for pebble, *calculus* (meaning "small stone"), gave him a verb, *calculare* (meaning "to calculate or reckon").

It is from this same word, *calculus*, that we get our modern word, the calculus, referring to the branch of

mathematics to which you are introduced in this issue of PRACTICAL MATHEMATICS. We do much more than the simple "reckoning" which the Roman attempted with his pebbles, however.

The calculus, the subject to which the greater part of the present issue is devoted, studies the rate of change of various quantities. Its uses are many and its practical adaptations are such that almost everyone has need of them.

Because of the fact that the calculus is dependent on algebra and trigonometry for the derivation of some of its formulas, we find it necessary to defer our study of the subject until we have laid a good foundation in the earlier branches. Do not be alarmed, however; this does not mean that the calculus is more difficult than many phases of these other subjects. Indeed, many high-school textbooks in algebra now introduce some of the simple aspects of the calculus as a part of the regular work.

By the time that you have read through Mr. Harvey's introductory article, you will probably be asking yourself why your teachers did not introduce you to this phase of the calculus years ago. You will be discovering for yourself ways in which you could have made good use of the calculus in solving problems that are constantly arising in your own daily affairs.

You will then be in a proper frame of mind to appreciate the more detailed treatment of the calculus which Dr. Wiener presents. This is a subject which unfolds so rapidly that I feel we must caution you once again

to "make haste slowly". Be sure that you understand each step before you attempt to read ahead to the next one. Just as the small child, piling his building blocks, must see that each rests firmly on a solid base lest the whole tower topple, so you, in working along from one formula to another, must be certain that you have properly related each to those preceding.

The present issue carries you a bit further along the way with two other items of "unfinished business". We dropped our study of complex numbers a couple of issues ago because it was necessary that we gain additional knowledge before pursuing the subject further. Now that you have a knowledge of trigonometry, we advance to the graphing of complex numbers, thereby adding also to the knowledge of graphs which we began to acquire back in Issue Number Four. Extending our consideration now to include polar coördinates, we are the better able to appreciate the practical uses to which we shall eventually put these complex numbers.

In order to understand the calculus in all its manifestations, we shall want to make sure that we really understand the uses of fractional exponents. We began to discuss these in connection with the study of logarithms away back in Issue Number Two. Later, in the study of algebra, in Issues Three and Four, we made further reference to them. Now we shall intensify our understanding of this useful device. The article is tucked away in the back part of the present issue (occupying pages 498 and 499), but it should really be read along with the article on the calculus. At the point in the calculus where you find a need for it, somewhere about page 462, drop your study of the calculus and turn to this brief article for a review and extension of your present understanding of

exponents. You will thus have a better outlook on the development of the formulas in the calculus.

Schools and colleges throughout the country are "drafting" instructors from other fields of knowledge to help teach classes in mathematics. In every city and large town, evening classes for adults who wish to learn more about mathematics are being offered. In almost every instance, the cry goes up that more classes would be offered if more instructors were available.

In my first chat, I suggested to you that, without a proper foundation in arithmetic fundamentals, no mathematics course could possibly get off to a good start. I urged you then to pay particular attention to the early lessons, lest you find yourself, sooner or later, in deep water. At various intervals, I have repeated this suggestion. Perhaps this is the proper time to say it once again: in your eagerness to get at the subject-matter of the applied issues, you may find yourself beset with the temptation to go too rapidly through some of our issues, without giving yourself a real opportunity to "think through" the propositions and theorems and formulas. Take time now and then to review what has gone before. I have endeavored to assist you to make these reviews by reminding you every now and then of the pages in the earlier issues where certain matters were treated. I hope that you are turning back to these from time to time, whenever you find the need for refreshing your memory. It is because we are thus tying together the various branches of mathematics that my colleagues and I feel we are assisting you in your self-study program. Unless you do your part, however, our effort is of no avail.

Practically every publisher has attempted to offer his own solution for the difficulty. Textbooks in math-

ematics are making their appearance faster than any one man can hope to keep up with them. Most of these books, however, are designed for use in the classroom, where an instructor may help the student master the subject. Few of them are usable for self-study.

Mathematics, far from being a mysterious process, is actually a means of solving mysteries. By this time, we have been able to point out to you how the various fields of mathematics, each in turn, provide a key to unlocking certain mysteries. Before you have completed the study and practice on the several topics, you will be in a position to know which key you should use to unlock each door that bars you from successful solution to your own problems.

That reminds me—and you, too, I hope. Some issues back, I had a few words to say about the need for constant practice to make sure that you were "getting" the subject-matter about which my colleagues and I have been talking. Is PRACTICAL MATHEMATICS your daily companion? A single issue is easily carried about—in street-car or bus or subway train, in suit-case or knapsack—for perusal in those odd moments which most of us don't somehow always use to the best advantage. Half an hour a day spent in reading, half an hour that otherwise might be wasted, will put you well on the way to furthering your mathematical knowledge. Just putting these magazines by in the hope that some day you'll have time to get around to them won't help you much in your present efforts to help shorten the war. Just reading won't solve the problem, however. There's a bit of work to be done. Another half hour spent in some quiet spot where you can get out a bit of paper and whatever tools you need for the particular subject now at hand, working on the solutions to some of the

exercises, will show you just what you have gained thus far.

Just how much of PRACTICAL MATHEMATICS will be of immediate use to you in your present position depends, of course, upon the position which you are filling. Obviously, in the small scope of a 64-page issue, we find it impossible to give an exhaustive treatment of each branch of mathematics. From the start, we have kept in mind the needs of specific groups of people—men who are expecting to enter the armed services and men and women who want to put mathematics to work in the shop or the industrial plant.

These people find it necessary to master the elements of mathematics just as quickly as possible in order to be ready to perform the task which is awaiting them. If they had more time at their disposal, they could have recourse to more complete treatises on the various branches, spending a year or more in studying each specific field, just as the student in the liberal-arts college does in peace-time. Unfortunately, the war won't wait for people to "catch up" in this fashion.

To be of service under the stringent demands of the present moment, one must acquire the "tricks of the trade" without spending on the consideration of theory the time which might well be devoted—if it were available—to that study. We can only hope that our presentation in PRACTICAL MATHEMATICS will sharpen your interest in the subject and whet your appetite so that you will be impelled to continue your studies in this fascinating subject somewhat more at your leisure when the present emergency is over.

With the aid of more than a score of experts, we have combed the field, determining just how much theory you need in order to understand the practical applications. Like the sage

of old, we have believed that wisdom or knowledge is not enough, that, while getting knowledge, it is necessary to get also understanding. For that reason, we have digressed at times into concise statements of the underlying theories, but always with the forthcoming practical applications in mind.

There is room in mathematics for a lifetime of study. This short course, covered in 14 issues at 10-day intervals does not profess to "cover" the subject, but merely aims to help you

select for your immediate use the formulas and techniques which may make you better able to play your part in the present emergency.

Many of you have written to tell me of the help which PRACTICAL MATHEMATICS has already been. As we approach the applied issues (Issues Number Ten through Fourteen), I am hopeful that we shall come even nearer to achieving our aim and satisfying your needs.

R.S.K.

A B O U T O U R A U T H O R S

NOBERT WIENER was born in Columbia, Missouri, in 1894, but received most of his education in Massachusetts, graduating from the Ayer High School and proceeding to Tufts College, where he received the degree of Bachelor of Arts. After a year of study at Harvard, he spent a year at Cornell, then returning to Harvard for two additional years of graduate work. He received the degree of Master of Arts from Harvard in 1912 and that of Doctor of Philosophy from the same institution in 1913. During the next two years, he studied and traveled in Europe, spending some time at Cambridge University and at Göttingen, after which he pursued post-doctoral studies at Columbia University.

Dr. Wiener was a docent-lecturer at Harvard during 1915-16, an instructor at the University of Maine during 1916-17, and a staff writer for the *Encyclopædia Americana* during 1917-18. After a short flight into newspaper work, he joined the staff of Massachusetts Institute of Technology as an instructor in mathematics in 1919, became an Assistant Professor in 1924, an Associate Pro-

fessor in 1928, and a Professor in 1932. During 1931-32, he lectured at Cambridge, England, and, during 1935-36, he was a Visiting Professor at Tsing Hua University, China.

Harvard gave him the Bowdoin prize in 1914 and the American Mathematical Society bestowed upon him the Bocher prize in 1933. During 1926, as Guggenheim fellow, he studied at Göttingen and Copenhagen.

In service as a civilian computer at the Aberdeen (Maryland) Proving Grounds during the first World War, Dr. Wiener has again been called upon during World War II to assist the federal government with his mathematical abilities.

He is a member and was at one time Vice President of the American Mathematical Society. He is also a member of the National Academy of Sciences and of the London Mathematical Society.

Among the more important of his writings are: *The Fourier Integral and Certain of Its Applications*, *Harmonic Analysis in Complex Domain*, and a great number of articles published in mathematical journals.

The Calculus

COURSE 2 Practical Mathematics PART 8

• AN INTRODUCTION TO THE CALCULUS

By Herbert Harvey

WITH the calculus, we enter an entirely different branch of mathematics. It was invented by Newton and Leibniz as a concise way of solving problems involving the relative variations of connected quantities which they could not solve with the resources of algebra and geometry, or else could solve only by very cumbersome operations. The calculus is not easy to master, but, once mastered, it is easy to use.

HOW WE USE THE CALCULUS

A very simple relation among rates is presented by the electromotive force of a battery, resistance of an electric circuit, and the current which flows through the circuit. These are connected by the relation,

$$I = \frac{E}{R}, \quad \text{I}$$

where E = EMF, R = resistance, I = current.

Suppose, for example, a storage battery with a constant EMF of 6 volts delivers current to a signal lamp with a resistance including the circuit of 10 ohms. The current through the lamp will then be 0.600 amperes, or 600 milliamperes.

The brightness of such a light varies very sensitively with the current. Let us inquire how the current will vary if we vary slightly the resistance of the circuit.

From I we can compute easily the values of the current for different resistances. These are:

$$EMF = 6 \text{ volts}$$

RESISTANCE (ohms)	CURRENT (millamps)
8	750
9	667
10	600
11	545
12	500

In this table, we find the value of the current for any one of several different values of R . It does not, however, answer the question, as to the rate of change of the current with change of R .

The table gives some information on the point. Between 8 ohms and 12 ohms, a difference of 4 ohms, the change in the current is 250 milliamperes,

or an average change of 62.5 millamps per ohm. At $R=10$, however, this average value does not hold. When R changes from 9 to 10 ohms, the current falls from 667 to 600, or 67 millamps. When R increases from 10 to 11 ohms, the current decreases from 600 to 545, or 55 millamps.

All of these numbers, each of which represents in one way or another the rate of change of current per ohm, are different. The question still is: under the conditions that *EMF* is 6 volts, and at the resistance of 10 ohms, what is the rate of change of current in millamps per ohm? We think it is somewhere between 55 and 67, and probably pretty near to 62.5.

We gain further light on the question by recomputing the table around $R=10$, from equation I at smaller intervals of R . The result is:

$$\text{EMF}=6 \text{ volts}$$

RESISTANCE (ohms)	CURRENT (millamps)
9.8	612.24
9.9	606.06
10.0	600.00
10.1	594.06
10.2	588.24

The rate of change of current per ohm, figured from the small interval, 9.9 to 10.0 ohms, is 60.6; figured from the interval, 10.0 to 10.1 ohms, it is 59.4 millamps per ohm.

We still do not have agreement in the rate per ohm; but it is clear toward what rate we are tending, as we make the change small enough over which the rate is computed.

CHANGE IN R (ohms)	EMF=6 volts	R=10 ohms	RATE OF CHANGE IN CURRENT (millamps per ohm)
-1			67
-0.1			60.6
+0.1			59.4
+1			55

The rate is 60 millamps per ohm—not 62.5 (the average misled us). We conclude that, while the change is not exactly uniform over appreciable intervals of R , the instantaneous rate of change at $R=10$ is 60.

We notice that this limiting rate is a simpler number than the actual rates through which we approached it. We also notice that it apparently has some connection with the 6 volts of the problem.

If we want to satisfy ourselves that this last connection is more than an accident, we shall have to go through these operations for some other value of the resistance (say, for 8 ohms). The results are (and here we have put the rates of change right into the table next to the figures from which derived):

Interval: 1 volt			Interval: 0.1 volt			Interval: 0.01 volt		
RESIST- ANCE (ohms)	CUR- RENT (mil's)	RATE OF CHANGE (per ohm)	RESIST- ANCE (ohms)	CUR- RENT (mil's)	RATE OF CHANGE (per ohm)	RESIST- ANCE (ohms)	CUR- RENT (mil's)	RATE OF CHANGE (per ohm)
6	1000	143	7.8	769.23	97.4	7.98	751.8797	94.13
7	857	107	7.9	759.49	94.9	7.99	750.9384	93.84
8	750	83	8.0	750.00	92.6	8.00	750.0000	93.66
9	667	67	8.1	740.74	90.3	8.01	749.0634	93.44
10	600		8.2	731.71		8.02	748.1298	

The rate of change of the current at EMF 6 volts, resistance 8 ohms, is thus 93.75 milliamperes per ohm.

It is not immediately clear what relation this number has to the 6 volts or the 8 ohms. If you are exceptionally observant, you may notice that 0.09375, which is the number of amperes represented by 93.75 milliamperes, is the entry in the table of aliquot parts for $\frac{3}{32}$; and that $\frac{6}{64}$, which this fraction equals, is the EMF in our problem, divided by the square of the resistance in ohms. This is also true of the rate of 60 milliamperes per ohm which we obtained before—that is, 0.060 amperes per ohm, equal to $\frac{E}{R^2}$ for that case.

This speculation concerning the relation between the rates and the magnitudes of the problem could be tested out by trying it for a large number of pairs of values of E and R . The calculations, while perfectly direct, would be cumbersome. Eventually we should probably conclude that the expression, $\frac{E}{R^2}$, actually does express the instantaneous rate of change in the current, per unit change in E , in a circuit with resistance, R . This would, however, be a highly laborious and complicated way of arriving at such a result. Further, if we had a complex formula to deal with, instead of the simple relationship of current, resistance, and electromotive force (if, for instance, we had to determine in this way the relative rates of motion of a piston and a point on the circumference of the flywheel) the labor would be proportionately more immense.

It is clear that we need a better way of solving problems such as this. What we want is to arrive at the limiting ratio of the rates at which two varying quantities change when they are connected by the conditions of some specified problem. We want to be able to arrive at such a limiting value without needing to know all of the values which may surround or lead up to it. Finally, the method of solution should be suitable not merely for working numerical problems, but also for finding a general solution in terms of x and y , so that we shall not have to go through long numerical tasks over and over.

The calculus is the means of finding solutions to problems which cannot be solved in any other way, or which can be solved by other means only through approximations involving great labor. In the calculus, we use numbers, variables, and limits in a more intensive way than we used them in arithmetic, algebra, and geometry. We must form much clearer pictures of what these seemingly simple things are—numbers, variables, and limits—before we can subject them to the rigors of the calculus.

NUMBERS | In arithmetic, we dealt with definite, specific numbers. The kind and variety of numbers in arithmetic include 2, 3, 17, $\sqrt{15}$, π , 0.00036, etc. In algebra, we first extended this scale of numbers backwards to include negative numbers. Then we introduced a different kind of number.

In algebra, when we write a for the length of a rectangle, we are using an unspecified number. Numbers in algebra may be of different degrees of generality: we may have a meaning the side of a rectangle, or x meaning the time when the minute hand crosses the hour hand between 1 and 2 o'clock (where x stands for an unknown definite number); or x_n , meaning any of the times when the minute hand crosses the hour hand (where x_n stands for twelve unknown but definite numbers); and so on. The algebraic symbols stand for particular, though not specific, numbers.

As we extend algebra, the degree of generality of the numbers increases. When, for example, we let c equal "any number", we still mean particular numbers. When, having derived the expression for the infinite series by which the numerical values of logarithms are computed, we wrote that

$$\log_e m = \log_e n + 2 \left[\frac{(m-n)}{(m+n)} + \frac{1(m-n)^3}{3(m+n)^3} + \frac{1(m-n)^5}{5(m+n)^5} \dots \right] \quad \text{II}$$

for any positive values of m and n , we were really thinking in terms of a whole series of values of m . The symbol, m , here means more than merely "some number", and more than merely "any number". It can mean each of the whole series of numbers from 1 to as large as we please, at intervals from each other as small as we please. Equation II can be interpreted to mean the whole logarithm table.

We are thus approaching a still more general type of number. This type of number is the subject of the theory of functions and the operations of the calculus.

If m is the series of numbers for which logarithms appear in a table, then m stands successively for 1, 2, ... n , $(n+1)$, $(n+2)$, ... up to the limit of the table, at intervals of 1. Again, a movie camera may print, every $\frac{1}{1000}$ th of a second, the position of a falling body. Then if t represents the number of seconds elapsed from the beginning of the film, we have t standing for a set of numbers 0.001, 0.002, ... 12.998, 12.999, 13.000, 13.001 Here t may cover the same range of numbers as m did, but at much smaller intervals. Each is still a series of *discrete* (individually distinct) numbers.

Variables

If we imagine the interval in such a series of numbers to become less and less, indefinitely, we finally get a series in which the individual terms are inseparable from one another; we approach a generalized number which represents, one after another, *all* the values between the initial and terminal boundaries. Such a number is called a *variable*.

In arithmetic, then, each symbol stands for a single, specified number. In elementary algebra, the symbols stand for discrete un-

specified numbers. In the calculus and after, we shall be dealing with variable numbers. A variable number embraces, one at a time, the entire range of values which a specified quantity can take on, between the limits or *bounds* of its possible values.

Numbers of the kind used in arithmetic and elementary algebra are called *constants* in discussions involving variables.

FUNCTIONS There is no great change made in the conclusions we reached in elementary algebra when we generalize certain of the numbers into variables. There is no loss, but some gain. For example, in algebra, we worked with the equation,

$$s = \frac{1}{2}gt^2, \quad \text{III}$$

where s is the distance in feet traversed by a body falling from rest, t is the time in seconds, and g is the constant number, 32.16. We said that equation III holds for any number, t .

In terms of variables, equation III means something like this:

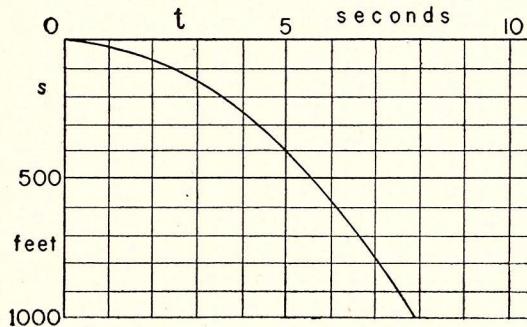


Fig. 1

This curve actually occurs. If a ball were thrown horizontally from the top of a building at a speed of 200 feet a second, the path followed by the ball would be this curve.

The distinction lies not in the substitution of a chart for an equation, but in the completeness of the second relationship against the discreteness of the first: we have extended III to cover *all* values of t over a particular range. The whole relationship between the two variables is shown in the second instance; only relations between particular pairs is shown in the first. The same expression governs both relationships. Until now, we have called the expression, $\frac{1}{2}gt^2$, a formula which gave the value of s for any particular value of t . In terms of variables, we shall call it a *function* of t designated by s .

Any expression involving a variable is a *function* of the variable. This general definition emphasizes the simplicity of the idea of function (which can be perplexing if we do not remember that mathematicians

have used the word in a special meaning which has practically no connection with its ordinary meaning in the language). Examples of functions:

$$s = \frac{1}{2}gt^2 \quad \text{Here } s \text{ is a function of } t. \quad \text{III}$$

$$y = x^3 - 5x^2 + 7x + 2 \quad \text{Here } y \text{ is a function of } x. \quad \text{IV}$$

$$z = \sin y \quad \text{Here } z \text{ is a function of } y. \quad \text{V}$$

$$p = q^2 - 2qr + r \quad \text{Here } p \text{ is a function of } q \text{ and also a function of } r. \quad \text{VI}$$

When we wish to write the statement that y is a function of x , without, however, specifying *what* function, we write $y = f(x)$, read "y equals the f -function of x " (or, more briefly, "y equals f of x "). Thus in the foregoing illustrations,

$$\begin{aligned} s &= f(t) & \text{IIIa} \\ y &= \phi(x) & \text{IVa} \\ z &= z(y) & \text{Va} \\ p &= P(q, r) & \text{VIa} \end{aligned}$$

Notice that we did not write the first of these as $s = f(g, t)$, although the equation for s involved g . This is because g is a constant, not a variable. In this case, g is a *numerical* constant. Sometimes, a function involves constants which are not numerical constants, but may take on different values. This does not make them variables. For instance, in III, a change in the latitude will involve a change in the value of g ; still, g is a constant in the relation between the variables, s and t . That is, g remains constant through any one problem. Indeed, if we had a problem where g varied within the fall (such as the fall of the moon in its orbit), equation III would no longer represent the relation between s and t . Thus g , even if different values be admissible for it, plays the part of a constant in III as characteristically as s and t play the part of variables.

A symbol which represents a constant in one problem and a variable in another problem is properly called a *parameter*; a parameter is sometimes loosely called a constant.

In a relation like IIIa, stating that s is a function of t , we refer to t as the *independent* variable in the equation and to s as the *dependent* variable. This is a terminology of convenience. Equation III might be written as

$$t = \sqrt{\frac{2s}{g}} \quad \text{IIIb}$$

so that t is also a function of s . In IIIb, s is the independent variable. Dependent and independent variable are therefore distinguished in a particular form of writing a relationship between the two variables; the distinction is not between the variables themselves.

In the form, $f(x)$, the independent variable, x , is sometimes called the *argument* of the function, a terminology used especially in designating a particular value of "the argument".

There is a third way of writing the relation, III, which is a useful form in some problems. Solving for g ,

$$g = \frac{2s}{t^2}. \quad \text{IIIc}$$

This is called the *implicit form* of the equation, or relationship between the variables. (Note that this is the form of the equation Galileo must have used in originally determining the value of g in his famous experiments with falling bodies at the leaning tower of Pisa.)

When an equation between two variables is in the implicit form, neither variable is the independent variable; indeed, the equation does not explicitly state either variable as a function of the other. The general form of the foregoing statement is written

$$g = q(s, t) \quad (g = \text{constant})$$

and in this form we say that s is an *implicit* function of t (or t of s).

LIMITS | Sometimes, but not always, a dependent variable, like the independent variable of which it is a function, takes on all the values over the range of numbers. Sometimes, however, from the form of the problem or otherwise, the dependent variable behaves in a quite different way at or near certain values of the independent variable, and the dependent variable is affected by a limit in the given region. Just at these limits, where the variables behave in a peculiar way, are apt to occur the points which count in our problems. We have already seen this in one example. Limits must be handled with precision; but they need not be as difficult as they are sometimes made.

An illustration from physics

Radium disintegrates through an atomic process whereby its atoms lose electrons in collisions with each other. The wounded atom shoots away from the collision as an atom of something which is no longer radium. The rate at which this goes on is proportional to the number of atoms which remain as radium, and hence is a continually decreasing rate.

After half the radium has degenerated into "emanation", the rate of degeneration has also become half.

Radium C'' is an exceptionally active substance, and degenerates at the rate of half its mass in about $1\frac{1}{2}$ minutes. A gram of radium C'' , after $1\frac{1}{2}$ minutes, becomes $\frac{1}{2}$ gram, which then emanates half as fast as the gram.

After another $1\frac{1}{2}$ minutes, this will become $\frac{1}{4}$ gram, which will radiate $\frac{1}{4}$ as fast. After $4\frac{1}{2}$ minutes, this fraction will have become $\frac{1}{8}$; and so on. Continually diminishing its rate of emanation in proportion to what is left, the substance never degenerates wholly out of existence, but continues to waste away at the rate of half of itself every one and one-half minutes as long as

there is enough of it left to detect by our instruments. (Do not be alarmed for the conservation of matter; the lump itself remains, but becomes the relatively inert residuum, radium D .)

We may say, under these conditions, that, as time goes on, the mass of radium C'' in the sample approaches zero as a limit. It never reaches zero, however.

If we look at the mathematics of this relatively simple process, we find that there are two different ways of formulating it. First, suppose we put n for the number of "half-periods", as they are called, of one and one-half minutes each in the life history of the sample. Then the mass of radium C'' in an initial sample of 1 gram will be, at the end of n half-periods, $m = \left(\frac{1}{2}\right)^n$ grams.

Here n is a discrete number, which may assume any positive value, as large as we please. Then our conclusion reads that the mass, m , approaches zero as a limit as the number of half-periods, n , increases without limit. Here we are dealing with a discrete series of fractions, each one-half the preceding, and diminishing indefinitely by steps.

Second, suppose we put t for the time in hours, and m for the mass of what is left of the sample. Then $m=1$ at $t=0$. This time we have two variables, connected by the equations,

$$m = k^{-t},$$

where k is the mass left at the end of 1 hour. Here t does not assume a discrete series of values, but increases continuously and indefinitely. The larger t gets, the nearer m gets to 0. Not only this, but we may select any number as small as we please, without being able to find one smaller than m will become if we allow t to increase enough. Under these conditions, we may say that, as t increases without limit, the variable, m , approaches 0 as a limit.

A geometric illustration

In geometry we discussed the perimeter of a circle as the limit approached by the perimeter of an inscribed polygon when the number of sides increases.

Taking a circle on a diameter of 1, and inscribing a polygon of n sides, each side, h , its perimeter will be nh . Doubling the number of sides, the length of a side (Fig. 2) is the hypotenuse, h' , where

$$h'^2 = \frac{1}{4}h^2 + \left(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{4}h^2}\right)^2 = \frac{1}{2} + \frac{1}{2}\sqrt{1-h^2} \quad \text{VII}$$

and the perimeter is $2nh'$.

The hexagon, for example, has $n=6$, $h=\frac{1}{2}$, so that its perim-

eter is 3. Putting $h=\frac{1}{2}$ in equation VII, we find $h' = \sqrt{\frac{1}{2} - \frac{1}{4}\sqrt{3}} =$

0.25882, and the perimeter is $12 \times 0.25882 = 3.1059$. Using 0.25882 as a new value of h in VII, we can get the side of a 24-gon, which, multiplied by 24 will give its perimeter; and so on. You may work this out further, or take our word for it that the limit of the procedure, as it is continued indefinitely, is the perimeter,

$$3.1415926535897932\dots$$

This limit is our definition of the ratio, π .

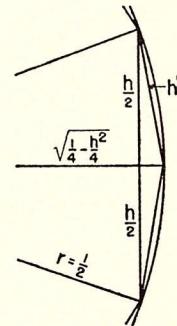


Fig. 2

Reaching a limit

We have been giving examples of limits which series or variables approach and *nearly but not quite* reach. A variable may approach a limit and reach it.

In Fig. 3, the left curve shows $\sin x$ approaching 1 as x approaches $\frac{\pi}{2}$; the right curve shows how $\frac{1}{x}$ approaches 0 as x increases without limit.

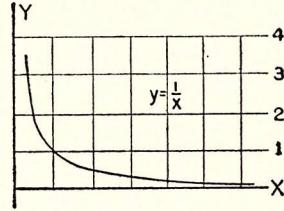
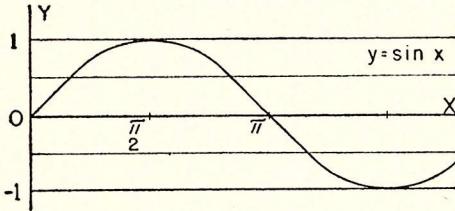


Fig. 3

The two cases are somewhat different. In either case, however, we may use the same notation,

$$\lim_{x \rightarrow \frac{\pi}{2}} \sin x = 1$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

(The symbolic statement, " $x \rightarrow \infty$ ", is read "x increases without limit" or "x becomes infinite".)

Continuous functions

Statements that series or variables approach limits are statements about infinitesimal magnitudes and should be made with precision. When so made, statements about limits can be extremely useful.

As an example, let us take the simple statement that, as x approaches c , a given function, $f(x)$, approaches its value when x is put equal to c —that is, approaches $f(c)$ as a limit (where c is a given constant). In the above notation,

$$\lim_{x \rightarrow c} f(x) = f(c).$$

VIII

One is tempted to say that this statement, which we have made about a particular function and a particular constant, is too obvious to state; and that it must be true for all variables and all constants. Let us go into this matter a little more carefully.

In Fig. 4, we have plotted the curve of $y = \tan x$ from $x = 0$ to $x = \pi$. (Compare with a table of tangents for 0° to 90° .) On this curve, look at the point

P which is at $x = \frac{\pi}{4}$, $y = 1$. At P , it is clear that equation VIII holds, with c put equal to $\frac{\pi}{4}$.

Now consider the case where $c = \frac{\pi}{2}$. The question is, does the value of $y = \tan x$ approach a limit as x approaches $\frac{\pi}{2}$? The answer is no. We cannot find a quantity, l , such that the difference between $\tan x$ and l can be made smaller than any assigned quantity by making x sufficiently near to $\frac{\pi}{2}$. In other words, there is no limit, and condition VIII is not fulfilled.

A curve is said to be *continuous* at a point, P , when condition VIII is satisfied at the point. Obviously, $\tan x$ is not continuous in the vicinity of $x = \frac{\pi}{2}$.

In higher mathematics, we have frequent occasion to know whether or not our problem is dealing with continuous curves and functions. The condition just noted is a precise way of stipulating what we mean by saying that a curve or function is continuous at a certain point.

To test the usefulness of this device, see if you can find a better, or, for that matter, any other precise way of saying what it means for a curve to be continuous at a point.

SUMMARY In the problem of the rate of change of an electric current, at the beginning of this introduction, we found that we were dealing with two *variables* and that one variable was a *function* of the other. We chose one as the *independent* variable and tried to determine what would be the ratio of a change in the *dependent* variable produced by a very slight change, or increment, in the independent variable. We found that the ratio came nearer and nearer to a certain value as the size of the increments became less and less. We concluded that the *limit* of this ratio as the size of the increments diminished indefinitely was a definite magnitude, and even concluded that, in terms of the original variables, the limit of the ratio was much simpler than the computations through which we approached it. If we had a mathematical process by means of which to arrive directly at the limiting ratio which was the answer to our problem, we could save much labor and time, and also obtain the result with considerable more certainty.

In the calculus, we create such mathematical processes. The calculus solves problems dealing with the relative rates of change of variable quantities connected by a functional relation. For most problems which the engineer and the mathematician encounter in practice, the calculus is necessary to arrive at a sufficiently accurate answer. In many, it is necessary if we are to arrive at any answer at all.

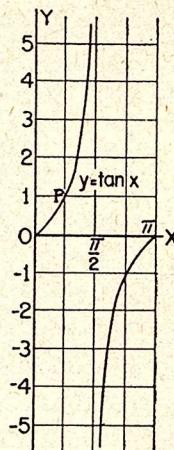


Fig. 4

• ELEMENTS OF THE CALCULUS •

By Norbert Wiener, Ph.D.

THE branch of mathematics we are now entering deals with a different order of problem than we have hitherto been studying. It is the only branch dignified in mathematical parlance by the definite article "the".

In entering the portals of the calculus, it is assumed we are equipped with an understanding of the fundamentals of what went before.

VARIABLES AND LIMITS Functions may involve one, two, or more variables. The range of a shell is a function of the angle of elevation of the gun, of the weight and temperature of the powder charge, the on-course and the cross-wind at all levels up to the top of the trajectory, the air-pressure at these levels, the latitude at which the gun is located, and of many other variables. The reading of a particular type of thermometer may be a function of the time. The stretch of a spring balance may be a function of the load. A function may (but need not) be represented by an explicit mathematical expression.

Functions

A function, $f(x)$ of x , is a way of assigning to every value of x over a given range a number, $f(x)$, of a specific kind. This number is the value of the function corresponding to the value of x .

In this article, we shall consider ranges of x which include only real numbers. We shall suppose that the values of $f(x)$ are also real.

Examples of functions of x are:

$$\begin{array}{c} x^2 \\ x^2 + x + 1 \\ (x)^a. \end{array}$$

A function, $f(x, y)$, of two variables, x and y , is a way of assigning to every pair of values of x and y , a number, $f(x, y)$, of a specific kind.

Examples of functions of two variables are:

$$\begin{array}{c} xy \\ x^2 + 2xy + y. \end{array}$$

An example of this is that the volume of a given mass of gas is a function of its temperature and pressure. If we say "z is the greater of the two quantities, x and y ", then z is a function of x and y .

In general practice, we use x , y , and z to represent the variables of which we take the functions. These are the *independent variables*. The letters, a , b , c . . . , are reserved for the constants of a problem. The letters, f , g , and the Greek letters, φ (read *phi*) and ψ (read *psi*), are reserved for functions.

The symbol, $f(x, y)$, is used to represent, not merely the form of the relationship involving the variables, but also to represent the value of the function for the pair of values, x, y , of the independent variables.

Illustrative Examples

A If

$$\begin{aligned}f(x) &= x^2 + 2x + 1 \\f(2) &= 2^2 + 2 \cdot 2 + 1 = 9 \\f(0) &= 0^2 + 2 \cdot 0 + 1 = 1 \\f(x+h) &= (x+h)^2 + 2(x+h) + 1\end{aligned}$$

B Another example, but this time with two variables.

If

$$f(x, y) = x^2 + xy + y^2$$

Then

$$\begin{aligned}f(0, 0) &= 0 \\f(x, 0) &= x^2 + x(0) + 0^2 = x^2 \\f(0, y) &= y^2 \\f(x, x) &= x^2 + x(x) + x^2 = 3x^2 \\f(x+h, y) &= (x+h)^2 + (x+h)y + y^2 \\f(x+h, y+k) &= (x+h)^2 + (x+h)(y+k) + (y+k)^2\end{aligned}$$

TEST YOUR KNOWLEDGE OF FUNCTIONS WITH THESE EXERCISES

1 $f(x) = x^3$. Find $f(x+h)$.

2 $f(x) = \log x$. Find $f(x+10) - f(x)$.

3 $f(x, y) = x^2y - y^2x$. Find $f(x, x)$.

4 $f(x) = \sin 2x$. Find $f\left(x + \frac{\pi}{6}\right) + f\left(x - \frac{\pi}{6}\right)$.

5 $f(x) = \cos 2x$. Find $f\left(x + \frac{\pi}{6}\right) + f\left(x - \frac{\pi}{6}\right)$.

Limits

In the introduction to this article, we said that, when a variable changes so that its value approaches a constant so closely that the difference between the constant and the variable becomes and remains less than any assigned positive number, the constant is said to be the limit of the variable. We also said if the variable changes in such a manner that it becomes and remains greater than any assigned positive number, it is said to become infinite.

With these facts in mind, we are ready to probe further into the field of limits. There are a number of important theorems concerning limits which we shall introduce at this point:

The limit of the sum of two variables, both of which have a limit, is the sum of their limits.

The limit of a constant, multiplied by a variable with a limit, is the constant times the limit.

The limit of the product of two variables, both of which have a limit, is the product of their limits.

If two variables approach limits, the second of which is not zero, then the quotient of the first divided by the second approaches the quotient of their limits.

If the limit of the divisor is zero, the quotient may become infinite, or it may have a definite value, but it is *not* determined by finding the quotient of the limits of the two variables.

I

II

III

IV

V

If the variable is represented by x and the constant by c , then we write the statement "x approaches c as a limit" in this manner:

"The limit of $f(x)$ as x approaches c as a limit is A " is written thus:

$$\lim_{x \rightarrow c} [f(x)] = A$$

To indicate that "x becomes infinite", we write:

$$x \rightarrow \infty$$

"The limit of $f(x)$ as x becomes infinite is A " is written thus:

$$\lim_{x \rightarrow \infty} [f(x)] = A$$

Illustrative Example

As an example to give us further insight into the subject of limits, and at the same time to assist us in future work, we shall show that:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Referring to Fig. 1, we note that, in a circle of unit radius, an angle of $2x$ radians is opposite a chord with a length of $2 \sin x$, an arc with a length of $2x$, and a segment of the tangent taken at the mid-point of this arc. Because the circle is of unit radius, the arc of the angle x is x (see Trigonometry, pages 388 and 389). The length of the segment of the tangent is $\tan x$.

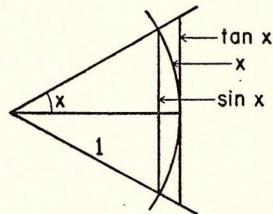


Fig. 5

Thus: $\sin x < x < \tan x$

If $x > 0$, and x is sufficiently near 0,

$$\frac{\sin x}{x} < 1$$

$$1 - |x| < \sqrt{1 - x^2} < \sqrt{1 - \sin^2 x}$$

VI

where $|x|$ means the "absolute" value of x (when x is positive, $|x|=x$; when x is negative, $|x|=-x$); or since

$$\sqrt{1 - \sin^2 x} = \cos x = \frac{\sin x}{\tan x}$$

$$1 - |x| < \frac{\sin x}{x}$$

VII

We have already indicated what we mean by saying that the limit of $f(x)$, as x tends to c , is A . Precisely it means that if δ is any positive quantity, then we can always find a positive quantity, ϵ , such that if $|x-c| < \epsilon$, then $|f(x) - A| < \delta$, whenever x and $f(x)$ are within the ranges allowed by the conditions of the problem.

In our problem, if δ is small enough, and $|x-0| < \delta$, equation VII becomes

$$\left| \frac{\sin x}{x} - 1 \right| < \delta$$

VIII

Thus, in the definition of limit, we can take δ itself for ϵ , and

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

IX

This simple derivation of a limit from the definition is characteristic

in method of less simple derivations to follow. Thoroughly assimilate each step.

DERIVATION OF e

A very important limit is that known as e . It is given by

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

To establish that a limit exists, let us consider the following inequality, which holds for all values of λ and x which are positive. It is

$$\lambda x < (1+x)^\lambda - 1$$

This formula may be verified by the use of the binomial theorem. If

$$(1+x)^\lambda = 1+y,$$

then

$$(1+y)^{\frac{1}{y}} = (1+x)^{\frac{\lambda}{y}} = (1+x)^{\frac{\lambda}{(1+x)^{\lambda}-1}} < (1+x)^{\frac{1}{x}}$$

That is, $(1+x)^{\frac{1}{x}}$ decreases with x , or $\left(1 + \frac{1}{x}\right)^x$ increases with x .

On the other hand, if n is an integer $>x$,

$$\begin{aligned} \left(1 + \frac{1}{x}\right)^x &< \left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + n\frac{(n-1)}{2}\left(\frac{1}{n^2}\right) + \dots + \frac{1}{n^n} \\ &< 1 + 1 + \frac{1}{2} + \dots + \frac{1}{2 \cdot 3 \dots n} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} \\ &= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} < 1 + \frac{1}{1 - \frac{1}{2}} = 3. \end{aligned}$$

Thus, as $x \rightarrow \infty$, $\left(1 + \frac{1}{x}\right)^x$ increases without ever exceeding 3. There is a theorem on limits to the effect that an increasing sequence which never exceeds a certain number has a limit itself not exceeding that number. Hence, e exists and does not exceed 3. Actually, it is 2.7182818....

So far, we have considered only $\left(1 + \frac{1}{x}\right)^x$ as x becomes infinite through positive values. Actually, this is not necessary, and

$$e = \lim \left(1 - \frac{1}{x}\right)^{-x}$$

as x becomes infinite through positive values. Let us put

$$u = x - 1.$$

Then

$$\left(1 - \frac{1}{x}\right)^{-x} = \left(\frac{1}{1 - \frac{1}{x}}\right)^x = \left(1 + \frac{1}{u}\right)^{1+u}$$

or

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^{-x} &= \lim_{u \rightarrow \infty} \left(1 + \frac{1}{u}\right)^{1+u} \\ &= \lim_{u \rightarrow \infty} \left(1 + \frac{1}{u}\right) \cdot \lim_{u \rightarrow \infty} \left(1 + \frac{1}{u}\right)^u = 1 \cdot e = e. \end{aligned}$$

Let us now consider the expression,

$$\begin{aligned} \frac{e^x - 1}{x} &= \frac{\left\{ \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y}\right)^y \right\}^x - 1}{x} \\ &= \lim_{y \rightarrow \infty} \frac{\left(1 + \frac{1}{y}\right)^{xy} - 1}{x}, \end{aligned}$$

since the function a^x is continuous in a for $x > 0$. By an inequality we have already given, this exceeds or equals

$$\frac{1}{x} xy \frac{1}{y} = 1.$$

On the other hand, if $x > 0$,

$$\begin{aligned} e^x &= \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n \rightarrow \infty} \left\{ 1 + x + n \left(\frac{n-1}{2}\right) \left(\frac{x}{n}\right)^2 + \dots + \left(\frac{x}{n}\right)^n \right\} \\ &\leq 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{4} + \frac{x^5}{4} + \frac{x^6}{8} + \frac{x^7}{8} + \dots \end{aligned}$$

or

$$e^x \leq \frac{1+x}{1-\frac{x^2}{2}}$$

so that

$$\frac{e^x - 1}{x} \leq \frac{\frac{1+x}{1-\frac{x^2}{2}} - 1}{x} = \frac{1+\frac{x}{2}}{1-\frac{x^2}{2}}$$

Hence, if $x \rightarrow 0$ through positive values, and $x^2 < 2$,

$$1 \leq \frac{e^x - 1}{x} \leq \frac{1+\frac{x}{2}}{1-\frac{x^2}{2}},$$

and

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

Again, if $x \rightarrow \infty$ through positive values,

$$\lim_{x \rightarrow 0} \frac{e^{-x} - 1}{-x} = \lim_{x \rightarrow 0} e^{-x} \frac{(e^x - 1)}{x} = 1 \cdot 1 = 1$$

Thus, in general,

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$$

The number, e , is used as the base of a system of logarithms, the so-called *natural logarithms*. We write

$$y = \log_e x \text{ for } x = e^y.$$

The *fundamental limit* theorem concerning natural logarithms is that

$$\lim_{x \rightarrow 0} \frac{1}{x} \log_e (1+x) = 1. \quad \text{X}$$

To prove this, let us put $1+x = e^u$. Then

$$\lim_{x \rightarrow 0} \frac{1}{x} \log_e (1+x) = \lim_{u \rightarrow 0} \frac{u}{e^u - 1} = \lim_{u \rightarrow 0} \frac{1}{\frac{e^u - 1}{u}} = \frac{1}{1} = 1$$

TEST YOUR KNOWLEDGE OF LIMITS WITH THESE EXERCISES

6 What is the limit as x approaches 1 of $\frac{1-x^2}{1-x}$?

7 What is $\lim_{x \rightarrow 0} \frac{\log_e (1+x)}{e^x - 1}$?

8 What is $\lim_{x \rightarrow 0} \frac{1-\cos x}{2-e^x-e^{-x}}$?

DERIVATIVES | In Fig. 6, you will note that point B is moving upward along a curve in the direction indicated by the arrow, upward and to the right. Taking any specific point on the curve, we may discuss the relationship between the rate of motion upward and the rate of motion to the right.

For instance, by inspection, we can see that, at point 1, B is moving upward about twice as rapidly as it is moving to the right. As it continues along its course through points 2 and 3, the motion upward decreases and the motion to the right becomes more apparent. At point 2, B is moving at about the same rate as it is moving to the right; at point 3, it has ceased to move upward at all. It is apparent that the ratio of the upward rate of motion to the rate of motion to the right is about 2 at point 1, 1 at 2, and 0 at point 3. It is possible to find, between points 1 and 3, a point where the ratio of the rate upward to the rate to the right has any value desired between 0 and 2.

Slopes

It is very important to note that the point, B , is moving in the direction of the tangent line to the curve, for any position of point B .

For example, when B is at point P , it is moving in the direction of tangent ST . The ratio of the upward rate to the rate to the right is equal to the tangent of the angle A , formed by the tangent line and the axis. We may call the trigonometric function $\tan A$ the *slope* of the tangent line.

The direction along a curve at any point is exactly determined by the *derivative*, which we shall now discuss at length.

THE DERIVATIVE AS A LIMIT

If $y=f(x)$, the *derivative*, or differential coefficient, is defined as the limit approached by the quotient of Δy by Δx as Δx approaches zero. Δx (read delta x) is a single symbol that stands for a small increase in the value of x and should not be construed to mean $\Delta \cdot x$. Δy stands for the corresponding increase in the value of y . Thus, when x increases to $x+\Delta x$, y increases to $y+\Delta y$, as in this example.

$$\begin{aligned}y &= f(x) \\y + \Delta y &= f(x + \Delta x)\end{aligned}$$

Before proceeding further, let us consider Fig. 7.

The curve, PQ , has the equation $y=f(x)$, and the line, PQ , is a secant which cuts this curve in P and Q and meets the X -axis at R . By recalling our method of reading graphs, we find that, at P , $x=OL$, and $y=PL$. Since $y=f(x)$, we may say $PL=f(x)$. Similarly, if $PN=h$, we have for the coördinates of Q , $x+h$ and $f(x+h)$; therefore

$$\begin{aligned}QN &= QM - PL = f(x+h) - f(x) \\ \tan QPN &= \tan QRX = \frac{QN}{PN} = \frac{f(x+h) - f(x)}{h}\end{aligned}$$

This quantity is called the *average slope* of the curve, PQ , over the interval, PQ .

Suppose we now let Q tend to P . In certain cases, this average slope will have a limit. This limit is called the slope of $y=f(x)$ at the point, P .

THE DERIVATIVE AS A SLOPE

If in the last equations, we put

$$h = \Delta x$$

then

$$f(x+h) - f(x) = \Delta y$$

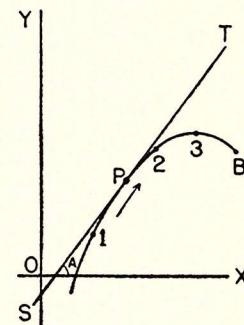


Fig. 6

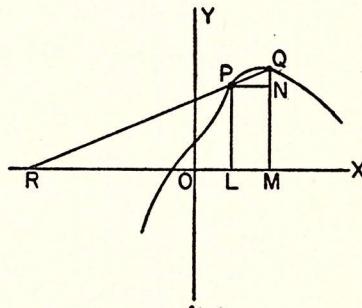


Fig. 7

and the slope

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

This is the definition of the derivative of $f(x)$ with respect to x . It is a function of x determined at P , the point at which it is taken.

The derivative is written $\frac{df(x)}{dx}$ or simply $\frac{dy}{dx}$. Thus,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

Here the d 's, like the Δ 's, are not symbols of multiplication; the entire symbol, $\frac{d}{dx}$, applied to a function, $f(x)$, means to obtain the derivative of the function, or to *differentiate* it. The notation for the derivative, $\frac{dy}{dx}$, is read, "the differential coefficient of y with respect to x ", or "the derivative of y with respect to x ". Sometimes we write y' for $\frac{dy}{dx}$.

Standard forms

Let us look at a few illustrative problems. The simplest derivative to take is that of x^n , where n is a positive integer. Using the binomial theorem, we have:

$$\begin{aligned} \frac{dx^n}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \rightarrow 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + h^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \dots + h^{n-1} \right) = nx^{n-1} \end{aligned} \quad \text{XI}$$

Certain non-algebraic derivatives are of importance. We have, using limit theorem IV,

$$\begin{aligned} \frac{de^x}{dx} &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = e^x \lim_{h \rightarrow 0} \frac{h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots}{h} \\ &= e^x \lim_{h \rightarrow 0} \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \dots \right) \\ &= e^x \end{aligned} \quad \text{XII.}$$

Similarly, by using X

$$\begin{aligned} \frac{d(\log_e x)}{dx} &= \lim_{h \rightarrow 0} \frac{\log_e(x+h) - \log_e x}{h} = \lim_{h \rightarrow 0} \frac{\log_e \left(1 + \frac{h}{x}\right)}{h} \\ &= \frac{1}{x} \lim_{h \rightarrow 0} \frac{\log_e \left(1 + \frac{h}{x}\right)}{\frac{h}{x}} = \frac{1}{x} \end{aligned} \quad \text{XIII}$$

By using limit theorem III, we can obtain

$$\frac{d \sin x}{dx} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{2 \cos\left(x + \frac{h}{2}\right) \sin \frac{h}{2}}{h}$$

(see page 425).

Here we use the fact that the cosine function is continuous, or that its limit for a given argument is equal to its value for that argument. Hence, using IX

$$\frac{d \sin x}{dx} = \cos x \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} = \cos x \quad \text{XIV}$$

It follows that

$$\begin{aligned} \frac{d \cos x}{dx} &= \frac{d \sin\left(x + \frac{\pi}{2}\right)}{d\left(x + \frac{\pi}{2}\right)} = \cos\left(x + \frac{\pi}{2}\right) \\ &= -\sin x \end{aligned} \quad \text{XV}$$

We obtained this derivative by putting $x + \frac{\pi}{2}$ as the independent variable in XIV; we could have derived it from the basic trigonometric formulas, as we derived XIV.

Illustrative Examples

A $y = ax + b$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{a(x + \Delta x) + b - (ax + b)}{\Delta x} = a$$

B $y = ax^2$, find $\frac{dy}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{a(x + \Delta x)^2 - ax^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2ax \Delta x + a(\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2ax + a \Delta x) = 2ax \end{aligned}$$

C $y = ax^3$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{a(x + \Delta x)^3 - ax^3}{\Delta x} = \lim_{\Delta x \rightarrow 0} \{3ax^2 + 3ax \Delta x + a(\Delta x)^2\} = 3ax^2$$

Note that when Δx approaches zero both $3ax \Delta x$ and $a(\Delta x)^2$ vanish. It would be possible for us to continue and show that the derivative of

$$\begin{aligned} ax^4 &= 4ax^3 \\ ax^5 &= 5ax^4 \end{aligned}$$

To designate the value of $\frac{dy}{dx}$ for a particular value x_1 of x , we employ

the notation $\frac{dy}{dx} \Big|_{x=x_1}$.

Now let us employ this notation in a problem.

Suppose we let $y = x^2$, and we want to find $\frac{dy}{dx} \Big|_{x=2}$ and so determine the slope of the tangent to the curve, $y = x^2$, at the point, $(x = 2, y = 4)$.

Solution

a $y = x^2$
 b $x = 2, y = 4$

c We increase x to $x + \Delta x$, with the result:

$$\begin{aligned}y + \Delta y &= (2 + \Delta x)^2 \\&= 4 + 4\Delta x + (\Delta x)^2\end{aligned}$$

d At $x = 2$,

$$y = 4$$

e Subtracting

$$\Delta y = 4\Delta x + (\Delta x)^2$$

f Dividing both sides by Δx :

$$\frac{\Delta y}{\Delta x} = 4 + \Delta x$$

g Letting $x \Delta \rightarrow 0$ we get the final result:

$$\frac{dy}{dx} \Big|_{x=2} = 4$$

Therefore, the slope of the tangent to the curve, $y = x^2$, at the point where $x = 2$ and $y = 4$ is 4. This is the value of the derivative we obtain from XI by putting $n = 2$, $x = 2$.

Differentiation formulas

Certain general formulas concerning derivatives are of great importance. The actual work of determining the derivative of any function may be greatly shortened by their use. Before going into an exposition of these formulas, we need to consider two more theorems:

The derivative of a variable with respect to itself is 1. The equation $y = x$ is interpreted graphically by a straight line with a slope equal to 1 and inasmuch as $\frac{dy}{dx}$ is the slope of the curve at any point, we may conclude that $\Delta y = \Delta x$. In fact,

$$\frac{dy}{dx} = \lim_{x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1 \quad \text{(Note to student—for graphic discussion see page 474.)}$$

The derivative of a constant is 0. This may be written:

$$\frac{dc}{dx} = 0$$

In what follows, the letters, v , u , x , and y , are used to indicate variables, while a , c , and n are used to indicate constants.

DERIVATIVE OF A SUM

The derivative of the sum of any number of functions is equal to the sum of their derivatives. Thus:

When u and v are employed to denote functions of x , and $y = u + v$, we may let $x = x_1$.

Then

$$y_1 = u_1 + v_1$$

If x takes the increment, Δx , then:

$$y_1 + \Delta y = u_1 + \Delta u + v_1 + \Delta v$$

By subtraction, we derive the result:

$$\Delta y = \Delta u + \Delta v$$

Then dividing both sides of the equation by Δx :

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

Finally, letting $\Delta x \rightarrow 0$, we have

$$\frac{dy}{dx} \Big|_{x=x_1} = \frac{du}{dx} \Big|_{x=x_1} + \frac{dv}{dx} \Big|_{x=x_1}$$

Generalizing these operations, we may arrive at a formula:

$$\frac{d(u+v+w+\dots)}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \dots$$

XVI

Illustrative Example

Find $\frac{dy}{dx}$ if $y = x^4 + 2x^2 - 3x + 4$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d(x^4)}{dx} + \frac{d(2x^2)}{dx} - \frac{d(3x)}{dx} + \frac{d(4)}{dx} \\ &= 4x^3 + 4x - 3.\end{aligned}$$

TEST YOUR KNOWLEDGE OF DERIVATIVES OF SUMS WITH THESE EXERCISES

In the following, find $\frac{dy}{dx}$:

9 $y = x^3 + 2x^2 + 2$

10 $y = 3x^2 - x + 6$

11 $y = x^3 + 3x^2 - x + 4$

12 $y = \sin x + \cos x$

DERIVATIVES OF A PRODUCT

The derivative of the product of two functions equals the first multiplied by the derivative of the second, plus the second multiplied by the derivative of the first.

In this instance, we let $y = uv$, and, as in the previous differentiation, $x = x_1$

Then:

$$y_1 = u_1 v_1$$

If x takes the increment Δx , then:

$$y_1 + \Delta y = (u_1 + \Delta u) (v_1 + \Delta v)$$

$$\Delta y = u_1 \Delta v + v_1 \Delta u + \Delta u \cdot \Delta v$$

Dividing both sides of the equation by Δx :

$$\frac{\Delta y}{\Delta x} = u_1 \frac{\Delta v}{\Delta x} + v_1 \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$

Finally, letting $\Delta x \rightarrow 0$, we have:

$$\frac{dy}{dx} \Big|_{x=x_1} = u_1 \frac{dv}{dx} \Big|_{x=x_1} + v_1 \frac{du}{dx} \Big|_{x=x_1}$$

We may generalize these operations in the formula:

$$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

XVII

We immediately conclude that the derivative of the product of a constant and a function equals the constant multiplied by the derivative of the function.

Putting c for the constant, write $y = cu$. Applying XVII, we get

$$\frac{dy}{dx} = c \frac{du}{dx} + u \frac{dc}{dx}$$

However, we also recall that

$$\frac{dc}{dx} = 0$$

Therefore, we write the general formula as:

$$\frac{d(cu)}{dx} = c \frac{du}{dx}$$

XVIII

Illustrative Problem A

If $y = (x-5)(x^2+3)$, find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= (x-5) \frac{d(x^2+3)}{dx} + (x^2+3) \frac{d(x-5)}{dx} \\ &= 2x(x-5) + (x^2+3) = 3x^2 - 10x + 3\end{aligned}$$

Illustrative Problem B

If $y = 2(x-4)^2$, find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = 2 \frac{d(x-4)^2}{dx} = 4x - 16$$

TEST YOUR KNOWLEDGE OF DERIVATIVES OF PRODUCTS WITH THESE EXERCISES

In the following problems, find $\frac{dy}{dx}$:

13 $y = 4(x-1)$

18 $y = 2 \cos x$

14 $y = (x^2+x)(3x-1)$

19 $y = (1-x)e^x$

15 $y = \frac{1}{2}(x+1)(x-1)$

20 $y = 2ax - ax^2$

16 $y = (x+5)(2x-2)$

21 $y = -(x-a)^2$

17 $y = (x^2+3x)(2x-x^2)$

22 $y = 4x^4 + 2x^3 + 3x^2 + 1$

DERIVATIVE OF A QUOTIENT

The derivative of the quotient of two functions is equal to the denominator multiplied by the derivative of the numerator minus the

numerator multiplied by the derivative of the denominator, all divided by the square of the denominator.

In this illustration, $y = \frac{u}{v}$. Then, as in the previous illustrations, when $x = x_1$, $y_1 = \frac{u_1}{v_1}$.

If x takes the increment, Δx , $x = x_1 + \Delta x$; then:

$$y_1 + \Delta y = \frac{u_1 + \Delta u}{v_1 + \Delta v}$$

Subtracting, we arrive at the result:

$$\Delta y = \frac{u_1 + \Delta u - u_1}{v_1 + \Delta v} - \frac{u_1}{v_1} = \frac{v_1 \Delta u - u_1 \Delta v}{v_1(v_1 + \Delta v)}.$$

Dividing both sides of the equation by Δx :

$$\frac{\Delta y}{\Delta x} = \frac{v_1 \frac{\Delta u}{\Delta x} - u_1 \frac{\Delta v}{\Delta x}}{v_1(v_1 + \Delta v)}.$$

Finally, we let $x \rightarrow 0$, then:

$$\left. \frac{dy}{dx} \right|_{x=x_1} = \frac{v_1 \left. \frac{du}{dx} \right|_{x=x_1} - u_1 \left. \frac{dv}{dx} \right|_{x=x_1}}{v_1^2}.$$

We may generalize these operations in the formula:

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

XIX

Illustrative Problem

If $y = \frac{x-2}{4x}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{4x \frac{d(x-2)}{dx} - (x-2) \frac{d(4x)}{dx}}{16x^2} = \frac{1}{2x^2}$$

TEST YOUR KNOWLEDGE OF DERIVATIVES OF QUOTIENTS WITH THESE EXERCISES

In the following problems, find $\frac{dy}{dx}$.

$$23 \quad y = \frac{x+2}{2x-1} \quad 24 \quad y = \frac{x}{x+1} \quad 25 \quad y = \frac{4x-3}{6x+2} \quad 26 \quad y = \frac{x-1}{1+x} \quad 27 \quad y = \frac{\sin x}{\cos x}$$

DERIVATIVE OF A POWER

The derivative of a function affected by a constant exponent, n , is equal to n multiplied by the function affected by the exponent, $n-1$, multiplied by the derivative of the function.

Taking n as a positive integer and $y = u^n$, and recalling the rule for the derivative of the product of two functions, we write $y = u \cdot u^{n-1}$; thus:

$$\frac{dy}{dx} = u^{n-1} \frac{du}{dx} + u \frac{d(u^{n-1})}{dx}.$$

If u^{n-1} is written as the product, $u \cdot u^{n-2}$, we get

$$\begin{aligned}\frac{dy}{dx} &= u^{n-1} \frac{du}{dx} + u \left[u^{n-2} \frac{du}{dx} + u \frac{d(u^{n-2})}{dx} \right] \\ &= 2u^{n-1} \frac{du}{dx} + u^2 \frac{d(u^{n-2})}{dx}.\end{aligned}$$

If this is repeated n times, the last term will contain $\frac{d(u^{n-n})}{dx}$, which, when we remember that the derivative of a constant is zero, is 0. Therefore, as a general statement, we may write the formula:

$$\frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx} \quad \text{XX}$$

Illustrative Problem

Find $\frac{dy}{dx}$, if $y = (x^2 + x + 2)^4$.

$$\frac{dy}{dx} = 4(x^2 + x + 2)^3 \frac{d(x^2 + x + 2)}{dx} = 4(2x+1)(x^2 + x + 2)^3.$$

TEST YOUR KNOWLEDGE OF DERIVATIVES OF POWERS WITH THESE EXERCISES

Find $\frac{dy}{dx}$:

28 $y = (x^2 + 3x + 4)^3$
29 $y = (2x^2 - 4x + 2)^6$

30 $y = (x^2 - 1)^4$
31 $y = (1+x)^{-2}$

32 $y = (2x^2 - 4x - 4)^2$
33 $y = \sin^2 x$

DIFFERENTIATING A FUNCTION OF A FUNCTION

A most important differentiation formula is that for a function of a function.

If $y = f(x)$ and $z = g(y)$, let a change of Δx in x generate a change of Δy in y , and let this generate a change of Δz in z . Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta z}{\Delta y} \frac{\Delta y}{\Delta x}$$

If $\frac{\Delta z}{\Delta y}$ and $\frac{\Delta y}{\Delta x}$ have limits $\frac{dz}{dy}$ and $\frac{dy}{dx}$, as Δx (and consequently Δy) approaches 0, we have, by III

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}. \quad \text{XXI}$$

For example, we know that

$$\sin^2 x + \cos^2 x = 1.$$

Differentiating,

$$\begin{aligned}0 &= \frac{d(1)}{dx} = \frac{d \sin^2 x}{dx} + \frac{d \cos^2 x}{dx} = \frac{d \sin^2 x}{d \sin x} \cos x + \frac{d \cos^2 x}{d \cos x} \frac{d \cos x}{dx} \\ &= 2 \sin x \cos x + 2 \cos x \frac{d \cos x}{dx}, \quad \text{whence } \frac{d \cos x}{dx} = -\sin x.\end{aligned}$$

Compare this with XIII.

Again, let $y = \log_e x$. Then

$$x = e^y$$

Differentiating,

$$\begin{aligned} 1 &= \frac{dx}{dx} = \frac{de^y}{dx} = \frac{de^y}{dy} \cdot \frac{dy}{dx} \\ &= e^y \frac{dy}{dx} = x \frac{dy}{dx} \end{aligned}$$

Since $y = \log_e x$

$$\frac{dy}{dx} = \frac{d(\log_e x)}{dy} = \frac{1}{x}$$

XXII

Similarly, let $y = \sin^{-1} x$. Then $x = \sin y$, and

$$1 = \frac{dx}{dy} = \frac{d \sin y}{dx} = \cos y \frac{dy}{dx} = \sqrt{1 - x^2} \frac{dy}{dx} = \sqrt{1 - x^2} \frac{d(\sin^{-1} x)}{dx}$$

or

$$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

XXIII

Since $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$,

$$\frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

XXIV

The expressions for the derivatives of the other inverse trigonometric functions may be similarly derived.

We have obtained, in expressions X to XXIV, the derivatives of the basic forms, that is, of the familiar functions we have encountered in our journeys through algebra, logarithms, and trigonometry, as well as the derivative of the sum, product, or quotient of two (or more) functions (XVI to XX); and, finally, the derivative of a function of a function (XXI).

These standard forms, both of the basic functions and of the function combinations, should be familiar to us—so familiar that we write them down at sight. Only in this way do we become able to obtain easily the derivatives of the complicated functions we find in everyday problems. These more complicated functions are usually built out of the basic forms. Their derivatives are therefore found by applications—if necessary, repeated applications—of the standard forms.

These forms are collected in Table XLI (page 512 of this issue). They include the derivatives of all the basic functions we have differentiated here, with the addition of some whose differentiation we have only indicated, as well as the formulas for differentiating the function combinations.

Differential notation

An important application of the principle that $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$ is that we may write $\frac{dy}{dx} = f(x)$ as $dy = f(x)dx$ without ambiguity, as $\frac{dy}{dz} = f(x) \frac{dx}{dz}$.

This notational device (for it is nothing more) is expressed by calling dx , dy , and dz *differentials*, and is known as the *differential notation*.

Illustrative Example

Write the expression for dy if $y = \sin^2 x$.

$$\begin{aligned}\frac{dy}{dx} &= 2 \sin x \frac{d(\sin x)}{dx} = 2 \sin x \cos x \\ &= \sin 2x \\ dy &= \sin 2x dx\end{aligned}$$

As exercises in the formula of differentiation given here, and in the differential notation, let us take up the following.

TEST YOUR KNOWLEDGE OF THE DIFFERENTIAL NOTATION WITH THESE EXERCISES

Find dy when:

34 $y = \sec x$ (use $\sec x = \frac{1}{\cos x}$)

37 $y = e^x$

35 $y = \tan x$ (use $\sec^2 x = 1 + \tan^2 x$)

38 If $y = f(x)$, then $y + \Delta y = ?$

36 $y = \tan^{-1} x$

39 If $y = ax^4$, find $\frac{dy}{dx}$

40 If $y = x^3$, find $\left. \frac{dy}{dx} \right|_{x=x_1}$ Draw the graph to determine the slope of the tangent to the curve, $y = x^3$, at the point ($x = 2, y = 8$).

41 Find $\frac{dy}{dx}$ (a) $y = \frac{x}{x^2+1}$, (b) $y = 5x^3 - 2x^2 + 3x - 6$,
(c) $y = (x^2+1)(2x+1)$

42 Give the functions for each of the following derivatives:

(a) $\frac{dy}{dx} = 1$ (b) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

See: Fractional Exponents (Page 499)

Special Graphic Illustrations of Derivatives

A In the preceding pages, we have already noted that the derivative of a variable with respect to itself is unity.
When $y = x$:

$$\frac{dy}{dx} = 1$$

Since $y = x$,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x}{\Delta x} = 1$$

We can present this graphically as in Fig. 8.

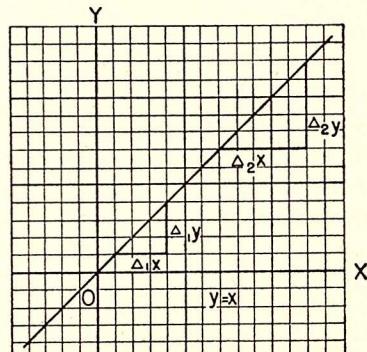


Fig. 8

B We have already discussed the subject of the derivative of a power. Now let us make a graphic representation of this.

Let $y = x^2$, and let $x = x_1$, therefore $y_1 = x_1^2$.

$$y_1 + \Delta y = (x_1 + \Delta x)^2 \\ = x_1^2 + 2x_1 \Delta x + \Delta x^2$$

Subtracting from this $y_1 = x_1^2$
 $y = 2x_1 \Delta x + \Delta x^2$

Dividing both sides by Δx we derive

$$\frac{\Delta y}{\Delta x} = 2x_1 + \Delta x$$

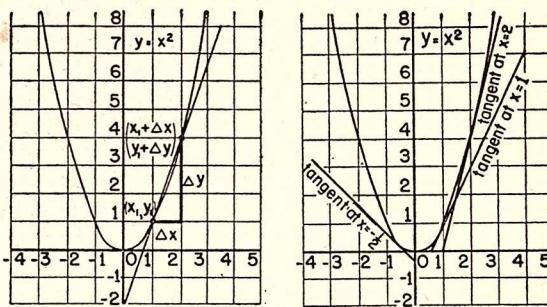


Fig. 9

Let $x \rightarrow 0$, then $\frac{dy}{dx} = 2x_1$ at any point, x_1 . Graphically we may represent this as in Figs. 9a and 9b.

When $x = 1$, the relative rate of change of the variables is 2 to 1.
 " $x = 2$, " " " " " " " " " " 4 to 1.
 " $x = -\frac{1}{2}$, " " " " " " " " " " -1 to 1.

C In the preceding section, we found that the derivative of the product of two functions equals the first multiplied by the derivative of the second, plus the second multiplied by the derivative of the first. Let us now demonstrate how this information might be used in an area problem. Suppose that we have a rectangle with variable sides and that we want to know the rate at which the area varies with respect to the time when the length of the sides are functions of the time. Remember that the length multiplied by the width gives the area of a rectangle.

Let A_1 = the area at a certain time, t_1

Let x_1 = the length of one side at a certain time, t_1

Let y_1 = the length of the other side at a certain time, t_1

To t_1 add a certain interval of time, Δt

At the time, $(t_1 + \Delta t)$, we have:

$$A_1 + \Delta A = (x_1 + \Delta x)(y_1 + \Delta y) = x_1 y_1 + x_1 \Delta y + y_1 \Delta x + \Delta x \Delta y$$

Subtracting:

$$\Delta A = x_1 \Delta y + y_1 \Delta x + \Delta x \Delta y$$

Each variable varies with respect to time, t ; therefore:

$$\frac{\Delta A}{\Delta t} = x_1 \frac{\Delta y}{\Delta t} + y_1 \frac{\Delta x}{\Delta t} + \Delta x \frac{\Delta y}{\Delta t}.$$

This result gives us the average rate at which the area of the rectangle varies with respect to the time as each side varies over the time interval, Δt .

The instantaneous rate of change of the area with respect to the time is the limit as $t \rightarrow 0$

$$\frac{dA}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}$$

for all values of x_1 or y_1 . This result may be stated as follows, "The in-

stantaneous rate of change of the area with respect to the time is the length multiplied by the rate of change of the width plus the width multiplied by the rate of change of the length."

HIGHER DERIVATIVES

The derivative of a function is itself a function, and as such its derivative may be determined. This derivative of a derivative is called the *second derivative* of the original function. The geometrical meaning of the derivative is a slope. When it is positive, the curve represented ascends from left to right. When it is negative, the curve descends from left to right. When it passes from negative to positive, the curve passes from a descending region to an ascending region, going through its lowest point in a given range. On the other hand, when the curve passes through a peak, the derivative passes from positive to negative.

If $y=f(x)$, then:

$$\frac{dy}{dx} = \frac{d\{f(x)\}}{dx},$$

Or as it may be written

$$\frac{d}{dx} f(x)$$

The second derivative of $f(x)$ is then $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, and is denoted by the symbol,
 $\frac{d^2y}{dx^2}$. XXV

In a similar manner, the *third derivative* may be written:

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

XXVI

In general, we write

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^n y}{dx^n}$$

XXVII

We call this the *nth derivative* of y with respect to x .

Illustrative Example

If

$$y = x^7 + \frac{1}{x^4},$$

$$\frac{dy}{dx} = 7x^6 - 4x^{-5} \text{ (Derivative)}$$

$$\frac{d^2y}{dx^2} = 42x^5 + 20x^{-6} \text{ (Second Derivative)}$$

$$\frac{d^3y}{dx^3} = 210x^4 - 120x^{-7} \text{ (Third Derivative), etc.}$$

Curves

Where a given curve represents y as a function of x (that is, y is the ordinate, x the abscissa), the representing curve will be of a form

well known to us if the function of x be either in the form

$$y = ax + b$$

XXVIII

or in the form

$$y = Ax^2 + Bx^2 + C$$

XXIX

where the coefficients are constants. The form XXVIII is called a *linear* function, because the "curve" representing it is always a straight line. The form XXIX, containing the square of the independent variable, is called a *quadratic* function (from the Latin word meaning square). The curve representing a quadratic equation is always a *conic section*. A conic section is a circle, parabola, ellipse, or hyperbola (or, in limiting cases, may reduce to a pair of straight lines or even a point). We are already familiar with the properties of conic sections (see pages 293 to 304).

Where the function is more complex in character than these, or where it is *transcendental* (meaning that it cannot be arranged as a series of integer powers of the independent variable), the curve representing the function will be of quite a different kind.

The first derivative of the function gives us the slope of the curve at any point. The second derivative of the function gives us the rate at which the first derivative is changing at that point. By evaluating these two derivatives at a few points (in addition to evaluating the function at these points) we can get a general picture of the curve without necessarily computing an immense number of ordinates and abscissae.

For instance, if at a point on a curve the second derivative is positive, it is clear the first derivative must be increasing. The slope of the tangent is *increasing* as we pass through such a point. It may be increasing from a small positive slope to a larger positive slope; then the curve is bending upward to the right. It may increase from a negative slope, through zero to a positive slope; the curve again is bending upward to the right. Finally, if the slope is increasing from

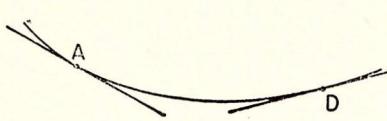


Fig. 10

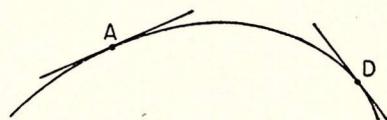


Fig. 11

a larger negative value to a lesser negative value, the curve is bending upward to the left.

In all cases, where the second derivative is positive, the curve is *concave*.

Similarly, when the second derivative is *negative*, at a given point, the curve is *convex* at that point.

Note in Fig. 10 that the slope increases from a negative value at the point of tangency A to a positive value at the point of tangency D , but in Fig. 11 the slope decreases from a positive value at point of tangency A to a negative value at D . The second derivative is *negative* and the curve is *convex*.

Illustrative Examples

A A parabola is represented by the equation $y=2-x^2$. Is the curve concave or convex at the point $x=0, y=2$?

The first derivative is $-2x$, and the second derivative -2 . Hence the curve is convex at the point (and at all its points). (Since we know what a parabola looks like, it is clear this particular parabola must be upside down. This we see is due to the fact that x^2 enters the function with a negative coefficient.)

B Is the curve representing $y=\sin x$ concave or convex at the point $x=90^\circ, y=1$?

The first derivative is $\cos x$, the second derivative, $-\sin x$, or -1 at the given point. The curve is convex there.

C Is the curve representing $y=(x-1)^3$ concave or convex at the following three points: $x=-1, y=-8; x=1, y=0; x=10, y=729$?

The second derivative is $6x$, which is negative at the first point, and positive at the second two points. The curve changes from concave to convex at the point where $x=0$, therefore has a convex portion and a concave portion, one of our points lying on the former, two on the latter. Draw the curve.

In the curve illustrated by Fig. 12, the tangents at various points have been drawn. At A , the slope is approaching 90° , at B , it is 45° , at C , it is 0° since there is no slope from the X -axis. Here we see that $\frac{dy}{dx}$ decreases from infinity below A to 1 at B ($\tan 45^\circ=1$), to 0 at C ($\tan 0^\circ=0$).

The graph of $\frac{dy}{dx}$, the slope of the curve in Fig. 12, is shown in Fig. 13. It decreases from infinity to 1 at B , 0 at C , and then becomes negative.

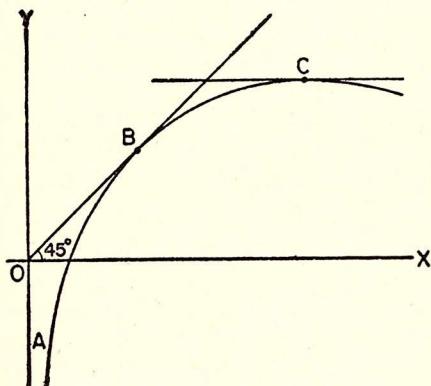


Fig. 12

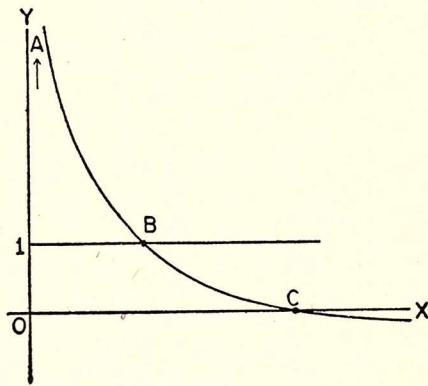


Fig. 13

Every tangent to this curve slopes upward to the left, and so the tangent of the angle of the slope is always negative. That is, $\frac{d^2y}{dx^2}$ is negative.

At this point, it is necessary for us to recognize several of the important characteristics of curves which will be repeatedly mentioned in the text material to follow.

Maxima and minima

In the curve represented in Fig. 14, note the points that are indicated, then consult the chart printed below the figure for the notations regarding these points.

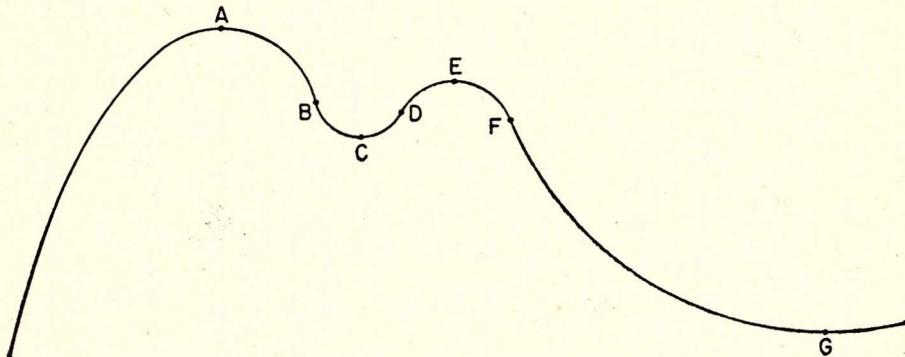


Fig. 14

Points *A* and *E*

These points are called *maximum points*, because the curve rises until it reaches these points, and then begins to fall. The value of the function at these points is called a *maximum ordinate*.

Points *C* and *G*

These points are called *minimum points*, because the curve falls until it reaches these points and then it begins to rise. The value of the function at such a point is called a *minimum ordinate*.

Points *B*, *D*, and *F*

At these points the concavity changes, and for this reason they are called *points of inflection*.

Still other facts must be indicated here. We distinguish five cases

A

$$\frac{dy}{dx} = 0; \frac{d^2y}{dx^2} > 0.$$

The curve is lower at x than at any other near-by point, and is said to have a *minimum*. (This is not necessarily the absolute minimum, where the curve reaches the lowest point of all.)

B

$$\frac{dy}{dx} = 0; \frac{d^2y}{dx^2} < 0.$$

The curve is higher at x than at any near-by point, and is said to have a *maximum*. This is not necessarily its absolute maximum.

C

$$\frac{dy}{dx} > 0 \text{ or } \frac{dy}{dx} < 0.$$

In this case, y is increasing or decreasing, and the given point cannot be either a maximum or a minimum.

D $\frac{dy}{dx} = 0; \frac{d^2y}{dx^2} = 0.$

The curve may have a maximum or a minimum or neither.

E $\frac{d^2y}{dx^2}$ changes sign at x .

The curve changes at x from being convex upward to being concave upward, and is said to have a point of inflection at x . It cannot have a maximum or a minimum at that point.

Illustrative Examples

Let us consider the curve $y = x^3 - x + 4$.

We have

$$\frac{dy}{dx} = 3x^2 - 1; \frac{d^2y}{dx^2} = 6x; \frac{d^3y}{dx^3} = 6; \frac{d^4y}{dx^4} = 0.$$

At the point, $x = \frac{1}{3}\sqrt{3}$, we find

$$\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$$

and the curve has a minimum.

Verify this by comparing the value of $f(x)$ for $x = \frac{1}{3}\sqrt{3}$, which is $f\left(\frac{1}{3}\sqrt{3}\right) = \frac{1}{9}\sqrt{3} - \frac{1}{3}\sqrt{3} + 4 = 3.6151$, with values of $f(x)$ for $x = 0.57, x = 0.58, x = 0.59$, neighboring to $x = \frac{1}{3}\sqrt{3}$. Values of y for x differing either way from $\frac{1}{3}\sqrt{3}$ are greater.

At the point, $x = -\frac{1}{3}\sqrt{3}$, we find

$$\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0,$$

and the curve has a maximum.

At the point, $x = 0$, we find

$$\frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} > 0.$$

Therefore, $\frac{d^2y}{dx^2}$ must change from a negative to a positive value at that point, and the curve has a point of inflection.

TEST YOUR KNOWLEDGE OF DERIVATIVES WITH THESE EXERCISES

Find the second derivatives of:

43 x^6

44 x^2

45 $ax^4 + bx^3 + cx^2$

46 Find the third derivative of $4x^3 + 3x^2 + 2x + 1$.

47 Find the third derivative of $\frac{1}{x^2}$.

48 How does the second derivative of x^3 differ from the third derivative of x^4 ?

The most definite practical consequence of this discussion of curves and their represented functions is the central fact that the derivative of a function vanishes at any point where the function has a maximum or a minimum value. At the point where it changes from a decreasing function to an increasing function, the function is a minimum; its instantaneous rate of change is zero and the second derivative is positive. At the point where it changes from an increasing to a decreasing function, the function has a maximum, the second derivative at such a point being negative.

We therefore locate the maximum and minimum values of any function (or curve) by differentiating the function, setting the derivative (which is an expression in the independent variable) equal to zero, and solve for the value of the independent variable. Substituting this value (or these values) of the independent variable in the function gives us the maximum and minimum values of the function. We may sort out the maxima from the minima by inspection, or by evaluating the second derivative of the function at each point.

This technique for the determination of maxima and minima has many practical applications.

Maximum and Minimum Problems

For example, if a given amount of power is to be transmitted over a power line, the resistance losses vary as the reciprocal of the voltage used, and the leakage losses increase directly as the voltage. The total losses may be written:

$$L = \frac{A}{v} + Bv$$

Here, $\frac{A}{v}$ represents the resistance loss, and Bv the leakage loss. The total loss is infinite if v is zero, or if v is infinite, and is finite for intermediate values. Therefore it must have some minimum point in between, and since it has a derivative at all intermediate points, the derivative must vanish there. To determine the points where the derivative vanishes, we write:

$$\frac{dL}{dv} = -\frac{A}{v^2} + B = 0.$$

The only admissible value of v that will satisfy the equation is the positive value, $v = \sqrt{\frac{A}{B}}$. This gives us $\frac{A}{v} = Bv$. Thus, we see that, for the most economical operation, the resistance losses should equal the leakage losses.

For our second problem, let us turn to ballistics.

The equations of motion of a projectile are known to be

$$x = v_0 t \cos \alpha + x_0,$$

$$y = v_0 t \sin \alpha - \frac{gt^2}{2} + y_0.$$

In these formulas, x and y represent the horizontal and vertical coördinates of the projectile, respectively; t represents the time; v_0 represents the muzzle velocity; g represents the acceleration of gravity; and α represents the angle of elevation of the gun.

If the projectile is fired from the gun and the elevation of the target is the same as that of the gun, we may indicate the time of flight as follows:

$$v_0 t \sin \alpha - \frac{gt^2}{2} = 0.$$

Since the solution, $t=0$, may be disregarded as referring to the gun itself, we use the solution

$$t = \frac{2v_0 \sin \alpha}{g}$$

The range is the horizontal distance from gun to target for this time, and is given by:

$$x - x_0 = \frac{2v_0^2 \sin \alpha \cos \alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}.$$

We now wish to determine the value of α , the angle of elevation, which will give us the greatest range, other things being equal:

$$0 = \frac{d}{d\alpha} \left(\frac{v_0^2 \sin 2\alpha}{g} \right) = \frac{2v_0^2 \cos 2\alpha}{g}.$$

Thus $\cos 2\alpha = 0$; then α must be between 0 and $\frac{\pi}{2}$, corresponding to zero range;

and α must be $\frac{\pi}{4}$ radians, or 45° . This is then the elevation of a gun which yields the greatest vacuum range. (It may be remarked that, under actual conditions, with air resistance playing a very important rôle, the angle is seldom if ever increased to much above 40° .)

For a third problem, let us consider the manufacture of wire. The cost of wire is made up of two parts: the cost of the material, and the cost of manufacture. The first cost is so much per pound, while the second cost is nearly directly proportional to the number of draws and annealings needed to reduce the section of the original billet to the final cross section. As the metal will not stand more than a certain amount of distortion per stage without annealing, the cost of manufacture is proportional to the negative logarithm of the final cross section.

Thus, the total cost of the wire manufactured from a given billet is expressed by

$$a - b \log_e x$$

where x is the cross section.

The cost per foot is then proportional to the total cost multiplied by the cross section, or

$$y = x (a - b \log_e x).$$

Let us now determine which section of wire will be cheapest and which

section of wire will be most expensive, per foot. If we differentiate, we get

$$\frac{dy}{dx} = (a - b \log_e x) \frac{dx}{dx} + x \frac{d}{dx} (a - b \log_e x) = a - b - b \log_e x;$$

$$\frac{d^2y}{dx^2} = \frac{b}{x}.$$

This indicates that there is no cheapest section (except $x=0$, for which the cost per foot will be 0). There is, however, a most expensive section, which we express as

$$\log_e x = \frac{a-b}{b}, \text{ and } x = e^{\frac{a-b}{b}};$$

and for which the total cost of the wire will be b . Whether this section is actually realizable, or whether the unrolled billet is most expensive per foot, depends upon the relative sizes of a , the total cost of the wire billet, and b , the cost of the wire. If b is less than a , it is clear that the most expensive wire per foot is the original billet itself.

Sometimes the derivative is more simply expressed, not as a function of the independent variable, but of the dependent variable, or of both (as where we differentiate implicit functions). In such a case, we save labor by eliminating one of the variables after differentiation.

One of the classic maximum-minimum problems is that of the design of a rectangular container with a square base, open on the top, which shall have a maximum volume for the amount of flat material used.

Here let the side of the square base be x , and the depth of the container be y .

The volume is x^2y , and the surface is x^2+4xy .

Since this is a constant, its derivative with respect to x is zero, and, since y is a function of x , we have:

$$2x+4y+4x \frac{dy}{dx}=0.$$

Again, since the volume is a maximum, its derivative with respect to x vanishes, and

$$2xy+x^2 \frac{dy}{dx}=0.$$

Eliminating $\frac{dy}{dx}$ from these two equations we get,

$$\frac{2x+4y}{4x}=\frac{2xy}{x^2} \text{ or } \frac{x}{2}+y=2y$$

Since $x=0$ is not an admissible answer, $y=\frac{x}{2}$, and the container is half as deep as it is wide and long.

It is frequently possible to shorten the work of determining maximum and minimum values by making use of the following facts:

If c is a constant, any value of x which makes $f(x)$ a maximum or a minimum makes $c+f(x)$ a maximum or a minimum.

Any value of x which makes $f(x)$ finite, but not zero and a maximum, makes $\frac{1}{f(x)}$ a minimum; and any value of x which makes $f(x)$ finite, but not zero, and a minimum, makes $\frac{1}{f(x)}$ a maximum.

XXXI

Any value of x which makes $f(x)$ a maximum or a minimum, and positive, also makes $\log f(x)$ a maximum or a minimum.

XXXII

If c is a positive constant, any value of x which makes $f(x)$ a maximum, or a minimum, also makes $cf(x)$ a maximum or a minimum.

If c is a negative constant, any value of x which makes $f(x)$ a maximum makes $cf(x)$ a minimum, and any value of x which makes $f(x)$ a minimum makes $cf(x)$ a maximum.

XXXIII

General rules for solving problems in maxima and minima

- Analyze the problem, determining what it is that is to be the maximum, and what it is that is to be the minimum. If the problem under analysis is in a field with which you are not familiar, verify that you have it formulated correctly before attempting to solve it.
- Let the maximum or the minimum desired be the function of y .
- Express y in terms of a single variable x .
- Determine the first derivative, and those values of x which make

$$\frac{dy}{dx} = 0.$$

- Determine the values of the second derivative for the points at which the first derivative is 0. This step will enable you to determine whether the function has a maximum or a minimum at each point under discussion.

TEST YOUR KNOWLEDGE OF MAXIMA AND MINIMA WITH THESE EXERCISES

- The product of two positive variables, x and y , is constant at c . What is the minimum value of their sum, s ? ($y = \frac{c}{x}$; $s = x + \frac{c}{x}$. Put $\frac{ds}{dx} = 0$.)
- A steel tank (cylindrical in shape) is to be made to hold 5000 cu. ft. of hydrogen. The steel is of uniform thickness. Find the dimensions of the tank which will make the surface area a minimum.
(Hint: Area of surface = $\frac{\pi D^2}{2} + \pi Dh$. Volume = $\frac{\pi D^2 h}{4}$. Differentiate surface with respect to diameter.)
- A tin can is made by stamping the rounded sides out as rectangular pieces of metal. The ends are also stamped out of sheets of metal, in such a way that the circular pieces of metal are tangent to one another, and form a hexagonal arrangement. Neglecting the scrap value of what remains after the ends are cut out, and also neglecting the seams, what are the most economical dimensions for a can of a given volume?
- What is the oblong of greatest perimeter that can be inscribed in a circle of radius 4"?
- A paper cup is formed from a sector of a circle by joining the edges, so that a cone is formed. What should be the dimensions of the cup for it to contain the greatest volume, given the radius of the paper blank?
- What are the conditions for maximum and minimum points on the curve of $y = \sin x \cos x$?
- A ship, sailing on a course 30° West of South, at 12 knots, sights a second ship 20 miles dead ahead, sailing due north at 10 knots. When will the second ship be nearest to the first?

**PARTIAL
DIFFERENTIATION**

A function $f(x, y)$ or $f(x, y, z)$ of two or more variables may be considered as a function of one of them—say x —with the other variable or variables as mere parameters. It may then be differentiated with respect to x , leaving the other variables unchanged. We write

$$\frac{\partial f(x, y)}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Thus

$$\begin{aligned} \frac{\partial (x^3y + y \sin x)}{\partial x} &= \lim_{h \rightarrow 0} \frac{(x+h)^3y + y \sin(x+h) - x^3y - y \sin x}{h} \\ &= 3x^2y + y \cos x. \end{aligned}$$

If a small change Δx be made in x , and a simultaneous small change Δy be made in y , the change in f which results will consist in $\frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + R$, where R is small in comparison with Δx , Δy , or both. We write this set of facts in the form

$$df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

XXXIV

Similar results hold for functions of more than two variables.

Illustrative Examples

A rectangle $3.1'' \times 5.2''$ has the approximate area

$$\{(xy)_{x=3, y=5} + 3 \cdot 0.2 + 5 \cdot 0.1\} \text{ sq. in.} = 16.1 \text{ sq. in.}$$

The precise answer is 16.12 sq. in.

Higher partial derivatives

The expressions $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ represent repeated differentiation with respect to x (or to y), and need no comment. By $\frac{\partial^2 f}{\partial x \partial y}$, we mean $\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$, or

$$\begin{aligned} &\lim_{k \rightarrow 0} \frac{\frac{\partial f(x+k, y)}{\partial y} - \frac{\partial f(x, y)}{\partial y}}{k} \\ &= \lim_{k \rightarrow 0} \frac{1}{k} \lim_{h \rightarrow 0} \left\{ \frac{f(x+k, y+h) - f(x+k, y)}{h} - \frac{f(x, y+h) - f(x, y)}{h} \right\} \\ &= \lim_{k \rightarrow 0} \lim_{h \rightarrow 0} \frac{f(x+k, y+h) - f(x+k, y) - f(x, y+h) + f(x, y)}{hk}. \end{aligned}$$

In case, as will very often be true,

$$\lim \frac{f(x+k, y+h) - f(x+k, y) - f(x, y+h) + f(x, y)}{hk}$$

exists as h and k tend to 0 independently, this limit will be both

$$\frac{\partial f}{\partial x} \left(\frac{\partial f}{\partial y} \right) \text{ and } \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right), \text{ and } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}. \quad \text{XXXV}$$

TEST YOUR KNOWLEDGE OF PARTIAL DIFFERENTIATION

56 A right circular cone has vertical angle 61° , and altitude 5.2 ft. By method of partial differentiation, find approximate volume, surface area, and slant height.

57 Check $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for $u = \tan^{-1} \frac{y}{x} + \sqrt{x^2 + y^2}$.

Graphical differentiation

When we graphically differentiate a given curve, we form a new curve by erecting ordinates determined from the slope of the given curve at several points. Let us graphically differentiate the curve shown in Fig. 15.

We begin by drawing tangent lines to the curve at points, A, B, C, D, E , and F , and then take the ratio of the vertical to the horizontal side of each triangle. This ratio will enable us to locate the derived curve.

In Fig. 15, the horizontal distance in each triangle is 1; therefore, the vertical rises are the derivatives; the ordinate of the derivative curve at $x=0$ is $a'A' = aa_1$; at $x=a'b'$, the ordinate is $b'B = bb_1$, etc.

Note that secant EF will approach the tangent at F if E is drawn nearer to F . Its slope is the average rate of change for the interval, MF . The slope is

$$\frac{EM}{MF} = \frac{EM}{e'f'} = q'Q',$$

whence

$$EM = (e'f') (q'Q').$$

This relationship shows us that the distance, EM , is equal to the area of strip $q'Q'$, by $e'f'$. In a similar manner, the areas of the adjoining strips equal eD , dC , and so on.

If the width of a strip is continually decreased or the number of strips is

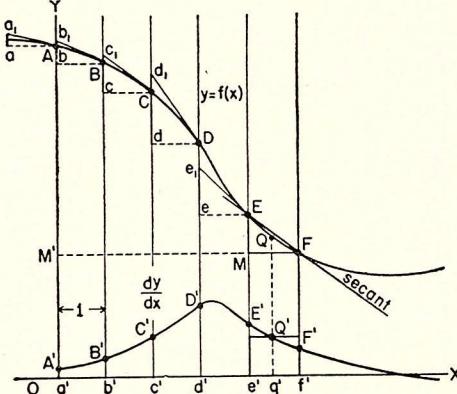


Fig. 15

continually increased, the sum of the partial ordinates will remain the length AM' in the primary curve, and the area of the strips will approach the area under the derived curve as a limit. The difference in length of ordinates of the primary curve equals the area between the ordinates, the derived curve, and the X -axis. From the difference in length of ordinates of the primary curve, areas of the individual strips are drawn, and an average curve drawn through the strips.

INTEGRAL CALCULUS | The inverse process to differentiation is called *integration*. Integral calculus is the inverse of differential. Its fundamental object is to find the function, the relation between the rates or differentials of the variables which enter it being given.

A function is called the *integral* of its differential, and the process by which we are going to derive it is called *integration*.

Differentiation, as its name implies, is closely related to the operation of taking differences.

Integration, on the other hand, is closely allied to the operation of taking sums. In fact, the symbol, \int , which is read "integral of", is nothing more than a long S-sign, and stands for the Latin word, "summa", or sum.

The process of integration is just the reverse of the process of differentiation, thus:

x^3 has as its differential $3x^2dx$

x^3 is the integral of $3x^2dx$.

In integral calculus, we write this as:

$$\int 3x^2dx = x^3,$$

read "the integral of $3x^2dx$ is x^3 ".

If G is any constant, we have

$$\frac{dC}{dx} = 0,$$

so that

$$\int 0 dx = C.$$

The integral of zero is an arbitrary constant, and the integral of the sum of two functions is the sum of their integrals; therefore, the integrals which we give are not fully determined functions, but are determinate except for the possibility of an indeterminate constant. Hence, two expressions for the integral of a function, obtained in

different ways, need not be exactly the same, but may differ by a constant. It is essential when integrating to add a constant, thus:

$$\int f'(x)dx = f(x) + C.$$

Fundamental formulas of integration

The integral of the sum of any number of differentials is the sum of their integrals. This follows from the rule that the derivative of a sum equals the sum of the derivatives (XVI).

Thus:

$$\int (du \pm dv) = \int du \pm \int dv. \quad \text{XXXVI}$$

The integral of a constant multiple of a variable is the constant multiplied by the integral of the variable.

Thus:

$$\int C du = C \int du. \quad \text{XXXVII}$$

The integral of a variable with a constant exponent in the differential of the variable is the variable with an exponent increased by one divided by the increased exponent.

Thus:

$$\int u^n du = \frac{u^{n+1}}{n+1} \quad (\text{Note: } n \text{ is not } -1) \quad \text{XXXVIII}$$

The integral of a fractional expression in which the numerator is the differential of the denominator is the natural log of the denominator.

Thus:

$$\int \frac{dx}{x} = \log_e x. \quad \text{XXXIX}$$

The integral of e with a variable exponent in the differential of the variable is e affected with the same exponent.

Thus:

$$\int e^u du = e^u. \quad \text{XL}$$

To find the integral of

$$a^u = e^{u \log_e a},$$

we use the principle that

$$\int u dy = \int u \frac{dy}{dx} dx, \quad \text{XLI}$$

which results from the fact that if $I = \int u dy$, then

$$\frac{dI}{dx} = \frac{dI}{dy} \frac{dy}{dx} = u \frac{dy}{dx}.$$

It follows that

$$\begin{aligned} \int a^u du &= \int e^{u \log_e a} du = \int \frac{e^{u \log_e a} d(u \log_e a)}{\log_e a} \\ &= \frac{e^{u \log_e a}}{\log_e a} = \frac{a^u}{\log_e a}. \end{aligned} \quad \text{XLII}$$

When a function to be integrated is exactly in the form of one of the fundamental formulas, the integral may at once be written by

that formula. If not, we try by transformation or substitution of variables to change it so that it shall be exactly like one of the fundamental integrals. In performing these operations considerable knowledge of trigonometry and algebra is frequently required. It is important to remember two facts, already mentioned:

The constant factor may be placed before or following the integral sign. The integral of the sum of any number of differentials is the sum of their integrals.

Illustrative Example A

Find $\int \frac{4x^4+21x^2-2x+20}{x^2+4} dx.$

By inspection of the above problem, and checking the list of formulas, we see that it can be reduced to basic forms thus:

$$\begin{aligned} \int \frac{4x^4+21x^2-2x+20}{x^2+4} dx &= \int \left(4x^2+5 - \frac{2x}{x^2+4} \right) dx \\ &= 4 \int x^2 dx + 5 \int dx - \int \frac{2x}{x^2+4} dx = \frac{4}{3} x^3 + 5x - \log_e(x^2+4). \end{aligned}$$

The last integral is evaluated from XXXIX, since $2x dx = d(x^2+4)$. In these solutions, we omit the constant of integration.

Illustrative Example B

Find $\int (a+b)^x dx.$

By inspection of the above problem, and checking the list of formulas, we note that we shall use formula LXII, thus:

$$\int (a+b)^x dx = \frac{(a+b)^x}{\log_e(a+b)}.$$

TEST YOUR KNOWLEDGE OF INTEGRATION WITH THESE QUESTIONS AND EXERCISES

What are the integrals of the following?

58 $x^4 dx$ 59 $(4x^3-2x^2+5x+6) dx$ 60 $x dx$ 61 $\sqrt{x} dx$

Find:

62 $\int \frac{dx}{x}$ 63 $\int (a+bx)^2 b dx$ 64 $\int v \frac{du}{dx} dx + \int u \frac{dv}{dx} dx$

65 What curve has an angle of slope whose tangent is $2x+1$?
(Hint: find $\int (2x+1) dx$)

Definite integrals

The usefulness of integration may be established by considering the geometric process leading to integration.

In Fig. 16, we see that OX and OY are the coördinate axes. CPD is a curve whose equation is $y=f(x)$. P is any point (x, y) on the curve. AC and PM are the ordinates at the points C and P , respectively. $F(x)$ is the algebraic area

between AB , the curve CP , PM , and MA . In other words any part of this area above the X -axis, and to the right of AB is positive. It is clear that the algebraic area $MPQN$ is $F(x+q) - F(x)$. It is also clear that the area, $MPQN$, must lie between the areas of the two rectangles, both with base MN , one of which has as its altitude the ordinate of the highest point in the interval PQ , while the other has as its altitude the lowest point. Calling the two altitudes H and h respectively, we have

$$h \leq \frac{F(x+q) - F(x)}{q} \leq H$$

If the function, $f(x)$, is continuous, we have

$$f(x) = \lim_{q \rightarrow 0} h = \lim_{q \rightarrow 0} H$$

$$f(x) = \lim_{q \rightarrow 0} \frac{F(x+q) - F(x)}{q}$$

$$f(x) = \frac{dF(x)}{dx}$$

$$F(x) = \int f(x) dx$$

The indeterminacy of this integral by an additive constant corresponds to the fact that the position of the line, AC , has not yet been specified. To overcome this difficulty, we introduce the definite integral. No matter what the constant of integration is, we put

$$\int_a^b f(x) dx = F(b) - F(a).$$

It will be seen that if we use the same value of the constant of integration (in other words the same function F) in the two integrals on the right-hand side of the expression, this constant will cancel out, and the expression will remain unchanged. Thus, if a is the abscissa of C , while b is the abscissa of D , the algebraic area, $ACDB$, will be found by substituting successively b and a in the indefinite integral, and subtracting. We write this operation

$$\int_a^b f(x) dx = \left[F(x) \right]_a^b = F(b) - F(a)$$

where $F(x)$ is the indefinite integral.

After discussing one or two other important factors, we shall take up the practical applications of integral calculus.

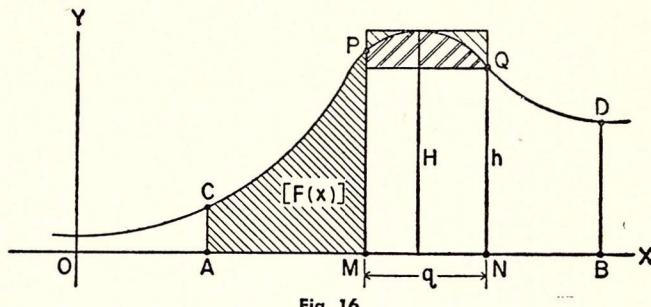


Fig. 16

Devices in integrating

There are certain tricks or devices that we can use in order to find integrals.

INTEGRATION BY PARTS

We know that:

$$\frac{d(uV)}{dx} = u \frac{dV}{dx} + \frac{du}{dx} V$$

By comparison, we get the integral equation:

$$uV = \int u \frac{dV}{dx} dx + \int \frac{du}{dx} V dx$$

XLIII

Therefore:

$$\int u \frac{dV}{dx} dx = uV - \int \frac{du}{dx} V dx$$

XLIV

We can put this expression in a more convenient form. $\frac{dV}{dx}$ is a function of x , which we shall call v . Then $V = \int v dx$. This gives us:

$$\int uv dx = u \int v dx - \int \left(\frac{du}{dx} \cdot \int v dx \right) dx.$$

XLV

Suppose we want to determine the area of the circle shown in Fig. 17.

$$x^2 + y^2 = a^2$$

$$y = \sqrt{a^2 - x^2}$$

The element of area is $y dx$ and we first have to integrate this between $x=0$ and $x=a$. We use the integral,

$$\int_0^a \sqrt{a^2 - x^2} dx = \left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= 0 + \frac{a^2}{2} \sin^{-1} \frac{a}{a} - \frac{a^2}{2} \sin^{-1} \frac{0}{a} = \frac{a^2 \pi}{2 \cdot 2}.$$

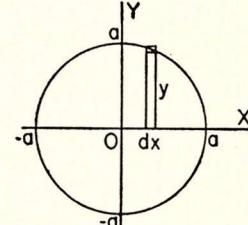


Fig. 17

Since we are working in radians, the angle whose sine is $\frac{a}{a} (=1)$ is $\frac{\pi}{2}=90^\circ$.

The area integrated is $\frac{1}{4}$ of the circle. The area of the whole circle is πa^2 , or π multiplied by the square on the radius.

INTEGRATION BY SUBSTITUTION

In many instances, the integral may be reduced to one of the fundamental forms by the substitution of a new variable.

Illustrative Example A

Find

$$\int \frac{2x^3 dx}{1+x^2}.$$

Put $y = 1+x^2$; then $dy = 2x dx$.

$$\begin{aligned}\int \frac{2x^3 dx}{1+x^2} &= \int \frac{y-1}{y} dy = \int dy - \int \frac{dy}{y} = y - \log y + c \\ &= C + x^2 - \log(1+x^2),\end{aligned}$$

where $C = c+1$ is the arbitrary constant.

If the integrand is rational except for a radical in the form $\sqrt{ax+b}$, it can be rationalized by substituting a new variable for the radical. If the integrand is rational, except for fractional exponents of x , it may be rationalized by substituting for x a new variable which is affected by an exponent equal to the least common multiple of the denominators of the exponents of x .

Illustrative Example B

Integrate

$$\int \frac{1+5\sqrt[3]{x}}{\sqrt{x}} dx = I.$$

Put $x = y^6$; $dx = 6y^5 dy$

$$\begin{aligned}\text{Then } I &= \int \frac{1+5y^2}{y^3} \cdot 6y^5 dy = 6 \int (y^2 + 5y^4) dy = 6 \left(\frac{y^3}{3} + y^5 \right) \\ &= 2\sqrt{x} + 6\sqrt[6]{x^5}, \text{ omitting arbitrary constant.}\end{aligned}$$

Sometimes ingenuity is required to find a substitution which, when found, will reduce an apparently unintegrable function to integrable form.

Illustrative Example C

Integrate

$$\int \frac{dx}{\sqrt{2x-x^2}}.$$

Substitute $x = 1-y$, $dx = -dy$, $2x-x^2 = 1-y^2$;

$$\frac{dx}{\sqrt{2x-x^2}} = -\frac{dy}{\sqrt{1-y^2}} = \cos^{-1} y = \cos^{-1}(1-x).$$

INTEGRATION BY THE USE OF A TABLE OF INTEGRALS

To facilitate the work of engineers and others who have frequent occasion to use integrals, tables have been compiled showing the integrals of forms frequently encountered in practical problems. These tables greatly reduce the work, since the simplest substitution will often reduce a given problem to a form found in the table (see Table XLII).

IMPORTANT APPLICATIONS OF CALCULUS

Speed and acceleration—The derivative will be found to have one of its most important uses in measuring of rates.

Illustrative Example A

A train starts from a station, R , and travels in a straight line at velocity v . After t seconds, it reaches a spot at a distance s from R .

The speed is represented as $\frac{ds}{dt}$, so that $\frac{ds}{dt} = v$. This is apparent, because a

small distance, Δs , is traversed in time, Δt . The speed equals distance divided by time, $\frac{\Delta s}{\Delta t}$, and the limit is $\frac{ds}{dt}$.

Acceleration is rate of change of velocity. Therefore acceleration is:

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = a$$

The second derivative represents acceleration.

We know that

$$\frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt}$$

Therefore:

$$\frac{dv}{dt} = \frac{dv}{ds} v$$

$$\frac{dv^2}{ds} = 2v \frac{dv}{ds}$$

$$\frac{dv}{ds} v = \frac{1}{2} \frac{dv^2}{ds} = \frac{d}{ds} \left(\frac{v^2}{2} \right) = \frac{dv}{dt}$$

Or, we may begin with

$$\frac{d^2s}{dt^2} = a,$$

LXVI

where a is a constant acceleration.

Integrate with respect to t :

$$\frac{ds}{dt} = at + u = v$$

LXVII

where u is an arbitrary constant added in integration.

When $t=0$, $u=v$. Thus, u is the initial velocity (at time 0), and

$$v = u + at$$

is the first equation of motion.

Now we integrate LXVII

$$s = \frac{1}{2} at^2 + ut + c.$$

When $t=0$, $s=0$ and $c=0$;

$$s = ut + \frac{1}{2} at^2$$

LXVIII

is the second equation of motion.

We have also seen that:

$$\frac{dv}{dt} = \frac{d}{ds} \left(\frac{v^2}{2} \right) = a$$

Integrating with respect to s :

$$\frac{1}{2} v^2 = as + d$$

When $s=0$, $v=u$ (the initial velocity), so that $d=\frac{1}{2}u^2$.

Hence,

$$v^2 = u^2 + 2as,$$

the third equation of motion.

When the motion is under gravity, we replace a by g .

Illustrative Example B

A bullet is fired in such a direction that it moves a horizontal distance in 10 seconds of $x = 500\sqrt{3} t$, and a vertical distance of $y = 500t - 16.1t^2$. Consider t as indicating time, and x and y as functions of t . Find the instantaneous velocity of the bullet at the end of 10 seconds.

$$\frac{dx}{dt} = 500\sqrt{3}$$

$$\frac{dy}{dt} = 500 - 32.2t$$

$$\frac{ds}{dt} = \sqrt{(500\sqrt{3})^2 + 178^2} = 884 \text{ feet per second}$$

Determining the volume and surface of a regular pyramid or cone—
In utilizing the calculus to determine measurements of regular pyramidal or conical objects, we may proceed as follows:

A To find the surface:

In Fig. 18, let P' = perimeter of base. $Oc = h'$ = slant height, P the perimeter of the element of area at d , $Od = s$.

Then

$$\frac{P}{P'} = \frac{Od}{Oc} = \frac{s}{h'}$$

Hence,

$$P = \frac{P's}{h'}$$

$$S = \frac{P'}{h'} \int_0^{h'} s ds$$

$$S = \frac{P'h'}{2}$$

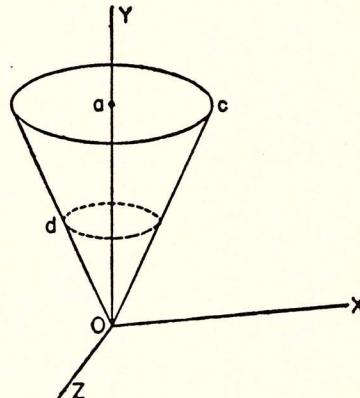


Fig. 18

We conclude that the convex surface of any pyramid or cone is measured by $\frac{1}{2}$ the product of perimeter of its base by its slant height.

B To find the volume:

In Fig. 18, let A' = area of base and $Oa = h$ = altitude. For the element of volume d at y , with area A ,

$$\frac{A}{A'} = \frac{y^2}{h^2} \quad A = \frac{A'}{h^2} y^2 \quad V = \frac{A'}{h^2} \int_0^h y^2 dy \quad V = \frac{A'h}{3}$$

Therefore, we may conclude that the volume of any cone or pyramid is measured by $\frac{1}{3}$ of the product of its base and altitude.

Determining the moment of inertia of a rectangle—The moment of inertia of any area about an axis in the plane of the area equals the integral of the product of the differential of the area and the square of its distance from the axis. The formula is:

$$M = \int r^2 dA$$

LXIX

A = area, r = distance of dA from the axis, M = moment of inertia. In a rectangle a wide and b long, let the X -axis bisect the width and the Y -axis bisect the length.

a If the axis of reference is OX

$$\begin{aligned} dA &= bdy \\ M &= \int_{-\frac{a}{2}}^{\frac{a}{2}} b y^2 dy = \frac{ba^3}{12} \\ &= \frac{Aa^2}{12} \end{aligned}$$

b If the axis is OY

$$\begin{aligned} dA &= ady \\ M &= \int_{-\frac{b}{2}}^{\frac{b}{2}} a x^2 dx = \frac{ab^3}{12} \\ &= \frac{Ab^2}{12} \end{aligned}$$

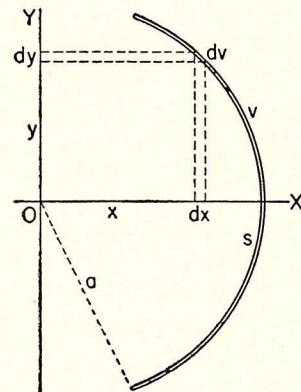


Fig. 19

Determining the center of gravity of a circular arc—The formulae for determining the center of gravity are:

$$x_1 = \frac{\int x dv}{v} \qquad y_1 = \frac{\int y dv}{v}$$

These formulae enable us to determine the coördinates (x_1, y_1) of the center of gravity of any given body of volume v . If the body is symmetrical with reference to a straight line, this may be taken as the X -axis, and $y_1 = 0$ on the X -axis.

In Fig. 19, arc s has as its axis of symmetry OX . We know that the center of gravity of a body is that point through which the line of action of the body's weight passes. Suppose we let x_0, y_0 be the coördinates of the extremity of the arc, s its length.

$$\begin{aligned} dv &= \sqrt{dx^2 + dy^2} \\ x_1 &= \frac{\int_{-y_0}^{y_0} x \sqrt{dx^2 + dy^2}}{s} \\ &= \frac{\int_{-y_0}^{y_0} x \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dy}{s} \end{aligned}$$

We know that the equation of the circle is $x^2 + y^2 = a^2$, so that

$$\frac{dx}{dy} = \frac{y}{x}$$

Therefore:

$$x_1 = \frac{\int_{-y_0}^{y_0} x \left[\frac{a^2}{x^2} \right]^{\frac{1}{2}} dy}{s} = \frac{2y_0 a}{s}$$

Since $2y_0$ is the chord of the arc, we can easily see that the center of gravity of a circular arc lies on the axis of symmetry of the arc and is located at a distance from its center equal to the fourth proportional between the arc, the radius, and the chord.

Determining the pressure of water on a surface—A physical principle, known generally as Pascal's Law, states that the pressure exerted upon a liquid in a closed vessel is transmitted by the liquid and distributed equally and undiminished in all directions. When we compute the pressure on a surface that is horizontal, we find that the pressure is equal to the weight of a column of the liquid standing on the surface as a base and of altitude equal to the distance the surface is beneath the surface of the liquid. To compute the pressure on a horizontal surface we use only simplest arithmetic. However, when we come to the problem of computing the pressure on a surface that is not horizontal, we find that we must employ principles of calculus, since in such instances the pressure varies with the distance beneath the surface.

For a *horizontal surface area*, S , at a distance a beneath the surface, P , when the density is w pounds, may be expressed by the formula, $P = waS$.

For a *vertical surface*, the pressure is found to increase with the depth, and the differential of the pressure is the differential of the area multiplied by wa , which may be expressed by the formula, $dP = wa dS$.

Illustrative Example

A circular gate 6 feet in diameter is used to close a water main. Find the pressure on the gate when the main is half full of water.

$$x^2 + y^2 = 9 \text{ (The equation of a circle)}$$

$$y = \sqrt{9 - x^2}$$

The liquid density $W = 62.5$ pounds.

Use the limits of integration $x = 3$, $x = 0$, covering half the area to be integrated. The fluid pressure is

$$62.5 \int_0^3 \sqrt{9 - x^2} x dx = \left[\frac{62.5}{3} (9 - x^2)^{\frac{3}{2}} \right]_0^3 = 562.5$$

The total pressure = $2 \cdot 562.5 = 1125$ pounds.

To compute problems involving work—It is apparent that, if we lift an automobile weighing 850 pounds a distance of 10 feet, we have a force of 850 pounds acting through a distance of 10 feet. When a force acts through a distance, the force is said to do work.

The unit of measure for work is the work necessary to lift one pound one foot. This unit is called a *foot-pound*. (In the metric system the unit of work done is the *gram-cm*.) A force of 1 pound acting through 5 feet does 5 ft.-lb. of work. Likewise, a force of 5 pounds, acting through 1 foot does 5 ft.-lb. of work. Therefore, we may say:

$$\text{work} = \text{average force} \cdot \text{distance}$$

In cases where the force is constant, simple arithmetic will give the answer. However, if it varies, it is necessary to revert to calculus. Two simple formulas will enable you to solve the problems in this category quite simply. In these formulas, F = force, c = constant, and s = distance.

When the force varies directly as the distance: $F = cs$. When the force varies inversely as a power of the distance: $F = \frac{c}{s^n}$.

Illustrative Example

A spring (coil of wire) obeys Hooke's law (that is, the force required to stretch it a small distance, x , is proportional to x , so that the work of stretching it a small distance is this force multiplied by x). If 10 pounds will stretch the spring 1 inch, what is the amount of work necessary to stretch it from 10 inches to 14 inches?

$$F = cs$$

$$c \cdot 1 = 10$$

$$c = 10$$

$$dW = F ds$$

This last statement is true, because the change in the work due to a change in the distance at any instant, gives the differential of the work.

$$dW = 10s ds$$

Integrating between the limits, $s = 2, 6$

$$W = \int_2^6 10s ds = \left[5s^2 \right]_2^6 \\ = 5(6^2 - 2^2) = 5(36 - 4) = 160$$

We were given 1 in. as the unit of measure. Therefore the answer in terms of foot-pounds is $\frac{160}{12} = 13\frac{1}{3}$ foot-pounds.

Additional applications of calculus to practical problems will appear in the issues of PRACTICAL MATHEMATICS which deal with the application of mathematics to machine-shop practice, heat and chemistry, navigation, and so forth.

FRACTIONAL EXPONENTS .

WHEN we were discussing logarithms, in Advanced Arithmetic, (page 90), we made a brief mention of fractional exponents. We found at that time that the logarithm is, in fact, an expression of a fractional or mixed-number exponent in decimal form.

Later, in our consideration of algebraic multiplication and division (for review, see pages 140 and 141), we learned that we could save ourselves many steps by combining exponents rather than multiplying out and then dividing long expressions. As we proceeded with our study of algebra, we found (turn back to pages 212 and 213) that fractional exponents, like integral exponents, made possible a simplification of our work.

In connection with our study of the calculus, we may wish to draw together what we have previously learned and to extend our concept of the fractional exponent as such.

FUNDAMENTAL LAWS OF EXPONENTS

First, then, let us refresh our memories as to the fundamental laws of exponents, as previously studied, expressing them in general terms which may be drawn upon as guiding formulas for any step we wish to take:

In multiplying two numbers having the same base, we write the base with the exponents added. (See page 141.) This may be expressed briefly as

$$a^m \cdot a^n = a^{m+n}. \quad \text{I}$$

To raise an exponential number to a higher power, we multiply the two exponents.

$$(a^m)^n = a^{mn}. \quad \text{II}$$

In dividing one number by another having the same base, we write the base with the exponents subtracted.

$$\frac{a^m}{a^n} = a^{m-n}. \quad \text{III}$$

To extract a root of an exponential number, we divide the original exponent by the index.

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}. \quad \text{IV}$$

When m and n are equal, we may substitute in III to obtain

$$\frac{a^m}{a^m} = a^{m-m} = a^0 = 1. \quad \text{V}$$

Likewise, when m and n are equal, IV becomes

$$\sqrt[m]{a^m} = a^{\frac{m}{m}} = a^1 = a. \quad \text{VI}$$

**WHEN EXPONENTS
ARE FRACTIONAL**

When the exponents are in fractional form, we have only to apply the same rules. Beginning with some numerical examples, in which we can see exactly what happens, we shall extend our observations to include generalized formulas for use in connection with any case.

Further consideration of IV gives us our clue. When we are confronted with a fractional exponent, $3^{\frac{2}{3}}$, we know, merely by reversing the terms in IV, that

$$3^{\frac{2}{3}} = \sqrt[3]{3^2}.$$

This means squaring the 3, getting 9, and then extracting the cube root (which may be done most simply by referring to a table of cube roots, such as that on page 251).

There is no essential difference between negative integral exponents and their counterparts among fractional exponents. As we have seen previously (page 142), $\frac{a^2}{a^4} = \frac{1}{a^2} = a^{2-4} = a^{-2}$, which may be expressed generally as $\frac{1}{a^m} = a^{-m}$.

VII

If the ratio between the two powers of the base is such that there is no common integral divisor, we should achieve a negative fractional exponent, such as $a^{-\frac{p}{q}}$.

This we should interpret as $\frac{1}{\sqrt[q]{a^p}}$.

Taking p and q to represent any incommensurable quantities, we should have as a general statement:

$$a^{-\frac{p}{q}} = \frac{1}{\sqrt[q]{a^p}}.$$

Illustrative Example A

Express the first four powers of 2 as fractional powers of 16.

$$16^{\frac{1}{4}} = 2.$$

$$16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} = (16^{\frac{1}{4}})^2 = 16^{\frac{1}{2}} = 2^2$$

and finally

$$16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} = (16^{\frac{1}{4}})^3 = 16^{\frac{3}{4}} = 2^3$$

$$16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} = (16^{\frac{1}{4}})^4 = 16 = 2^4$$

This last equality looks familiar to us, and verifies the rest.

Illustrative Example B

A piece of metal was cut to be exactly a 100-pound cube. The weight of the piece turned out to be 97 pounds. By what factor should the dimensions of the cube be increased?

The ratio of increase is

$$\left(\frac{100}{97}\right)^{\frac{1}{3}} = \left(\frac{1.00}{0.97}\right)^{\frac{1}{3}} = (0.97)^{-\frac{1}{3}}.$$

Expand this by the binomial theorem

$$(0.97)^{-\frac{1}{3}} = (1 - 0.03)^{-\frac{1}{3}} = 1 + \frac{1}{3} \cdot (0.03) + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!} (0.03)^2 + \dots$$

$$1.0000 + 0.0100 + 0.0002 + \dots = 1.0102 \dots$$

Thus the side of the cube must be increased by 1.02%.

• FURTHER STEPS WITH COMPLEX NUMBERS •

By Reginald Stevens Kimball, Ed.D.

When we first dealt with the imaginary number, expressed as i , (pages 358 to 367), we defined i as a number such that its square, i^2 , would equal -1 . It is possible to locate such a number on a graph, in much the same way that we have previously learned how to locate real numbers (pages 237 to 241).

PLOTTING IMAGINARIES

Again using quad-ruled cross-section paper, let us see how an imaginary number compares with a real number. In Fig. 20, we first locate 3 on the $X'X$ -axis (at point A). The quantity, -3 , would obviously be located, as indicated, at point B , since negative numbers are located to the left of the origin. Now the line-segment, OA , has rotated through an angle of 180° to the position represented by the line-segment, OB .

By our definition, it is obvious that the line, OB , also represents $3i^2$, since $i^2 = -1$ and $-3 = 3(-1)$. After taking several similar cases and finding that the negative number always lies on the line, $X'O$, we conclude that this portion of the $X'X$ -axis may be taken to represent i^2 . It follows immediately that OY , the perpendicular from the origin (since it lies halfway between OX and $X'O$), may be taken to represent i .

We have now found a way to represent imaginary numbers on the familiar graph paper. The point, C , on OY , since it is at a distance of 3 units from the origin, represents $3i$, just as A represents 3 and B represents -3 (or $3i^2$). Similarly, D represents $-3i$ (or $3i^3$). We do not need a special point to represent $3i^4$, since $i^4 = 1$ and $3i^4$, therefore, equals 3.1 (or 3, a real number).

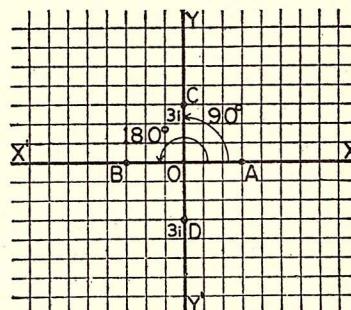


Fig. 20

TEST YOUR ABILITY TO PLOT IMAGINARIES GRAPHICALLY

Locate the following points in relation to the origin:

1 $4i$	4 $7i^3$	7 $5i^2$	10 $7\sqrt{-1}$	13 $0.5i$
2 $-7i$	5 $2i^4$	8 $-6i$	11 $-2\sqrt{-1}$	14 $-3.5i^2$
3 $6i$	6 $3i^5$	9 $2i$	12 $6i^6$	15 $2.25i^3$

Graphic representation of complex numbers

Carrying our thinking a step further, we are now ready to show on the graph the location of a complex number.

In such a combination as $5+2i$, we first locate the 5 on the line, OX ,

and the 2 on the line, OY , connecting these points with O by drawing straight lines. On the graph-grids which are perpendicular to the axes at the points located, we erect lines completing the rectangle and intersecting at P . The diagonal, OP , then represents the complex number, $5+2i$. Such a diagonal is called a *vector*, and is usually depicted with an arrow at the end, indicating direction.

Vectors are of great use in helping us to compute problems involving velocity, acceleration, gravitational attraction, stress and strain, etc. We shall employ them many times in connection with the issues on applied mathematics later in this course.

When we have occasion to add two forces involving complex numbers, we proceed in the same way, but secure a parallelogram (known as the *parallelogram of forces* or of *velocities*) instead of a rectangle.

Illustrative Examples

A Add graphically $4+7i$ and $6+3i$.

a Measuring 4 on OX and 7 on OY , we erect perpendiculars intersecting at the point, OP_1 . (Fig. 22.)

b Measuring 6 on OX and 3 on OY , we erect perpendiculars intersecting at the point, OP_2 .

c Draw P_1P_3 parallel to OP_2 and P_2P_3 parallel to OP_1 .

d The line, OP_3 , then represents the sum of the vectors, OP_1 and OP_2 , and therefore the value of $4+7i$ and $6+3i$.

B Subtract graphically $3+i$ from $5+4i$.

a Locate, as before, OP_1 , representing $5+4i$, and OP_2 , representing $3+i$. (Fig. 23.)

b Draw P_1P_3 parallel to OP_2 .

c Connect P_3 with O .

d The line, OP_3 , represents the difference of the vectors, OP_1 and OP_2 , and therefore the value of $(5+4i) - (3+i)$.

TEST YOUR ABILITY TO PLOT VECTORS GRAPHICALLY

Add graphically:

16 $3+3i$ and $5+2i$

17 $7+i$ and $2+6i$

18 $-3+2i$ and $4i-1$

Subtract graphically:

19 $1+i$ from $7i+4$

20 $2+3i$ from $4i-3$

21 $8i+3$ from $-4-2i$

POLAR FORM OF COMPLEX NUMBERS

Complex numbers may also be represented by polar coordinates, a form of representation in which the value of the angle formed by the vector plays a part. In Fig. 24, we have the point, P , representing the end of the vector, OP , with the coördinates, (x,y) ,

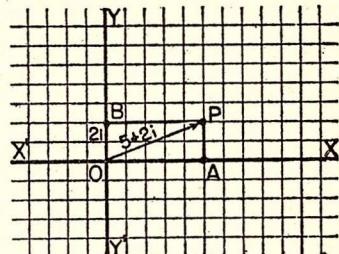


Fig. 21

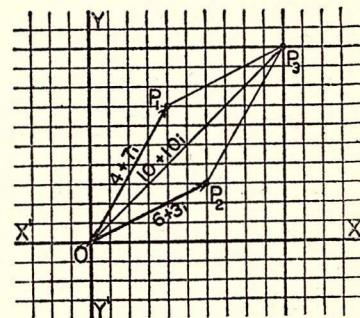


Fig. 22

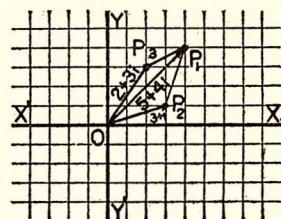


Fig. 23

as indicated. Taking the distance, OP , equal to ρ , and the angle XOP , equal to θ , we know immediately, from plane trigonometry, that

$$\begin{aligned}x &= \rho \cos \theta \\y &= \rho \sin \theta\end{aligned}$$

Substituting these in our equation, we get

$$x+iy = \rho (\cos \theta + i \sin \theta). \quad \text{I}$$

This is the *polar form* of the complex number. We call θ the *angle* or *argument* and the length of OP (ρ) the *modulus* or the *absolute value*. (Note that $\rho \geq 0$.)

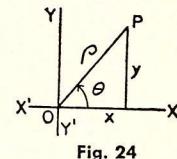


Fig. 24

TEST YOUR ABILITY TO CHANGE TO THE POLAR FORM

$$22 \ 4+4i \quad 23 \ -i+1 \quad 24 \ 12+5i \quad 25 \ i\sqrt{3}+1 \quad 26 \ 4i$$

Multiplication of complex numbers

The polar form is very helpful when it comes to the multiplication or division of complex numbers. By geometric interpretation,

$$\begin{aligned}(x_1+iy_1)(x_2+iy_2) &= x_1x_2+iy_1x_2+ix_1y_2+i^2y_1y_2 \\&= (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1).\end{aligned}$$

Since

$$x_1+iy_1 = \rho_1(\cos \theta_1 + i \sin \theta_1)$$

and

$$x_2+iy_2 = \rho_2(\cos \theta_2 + i \sin \theta_2),$$

we get the polar form,

$$\begin{aligned}(x_1+iy_1)(x_2+iy_2) &= \rho_1\rho_2[\cos \theta_1 \cos \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) - \sin \theta_1 \sin \theta_2] \\&= \rho_1\rho_2[\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2)]. \quad \text{II}\end{aligned}$$

Division of complex numbers

Similarly, in dividing, we arrive at

$$\begin{aligned}\frac{x_1+iy_1}{x_2+iy_2} &= \frac{\rho_1(\cos \theta_1 + i \sin \theta_1)}{\rho_2(\cos \theta_2 + i \sin \theta_2)} \cdot \frac{(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 - i \sin \theta_2)} \\&= \frac{\rho_1[\cos(\theta_1-\theta_2) + i \sin(\theta_1-\theta_2)]}{\rho_2(\cos^2 \theta_2 + \sin^2 \theta_2)} \\&= \frac{\rho_1}{\rho_2}[\cos(\theta_1-\theta_2) + i \sin(\theta_1-\theta_2)]. \quad \text{III}\end{aligned}$$

The expressions II and III for the product and quotient of complex numbers result in the celebrated theorem of De Moivre:

The absolute value of the product of any number of complex numbers equals the product of their absolute values.

The angle of the product of any number of complex numbers equals the sum of their angles.

Thus, the product of n equal factors gives:

$$[\rho(\cos \theta + i \sin \theta)]^n = \rho^n(\cos n\theta + i \sin n\theta).$$

IV

If $\rho=1$, this reduces to

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

V

If $n=-1$, we have

$$\begin{aligned} (\cos \theta + i \sin \theta)^{-1} &= \frac{1}{\cos \theta + i \sin \theta} = \frac{\cos \theta - i \sin \theta}{\cos^2 \theta + \sin^2 \theta} \\ &= \cos(-\theta) + i \sin(-\theta). \end{aligned}$$

VI

Extending this to consider n as any negative integer, we get

$$\begin{aligned} (\cos \theta + i \sin \theta)^{-p} &= [\cos(-\theta) + i \sin(-\theta)]^p \\ &= \cos(-p\theta) + i \sin(-p\theta). \end{aligned}$$

VII

Similarly it follows that, if n is fractional (represented by $n = \frac{p}{q}$),

$$(\cos \theta + i \sin \theta)^{\frac{p}{q}} = \left[\left(\cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right)^q \right]^{\frac{1}{q}} = \cos \frac{\theta}{q} + i \sin \frac{\theta}{q}$$

VIII

Combining V and VIII, we get

$$(\cos \theta + i \sin \theta)^{\frac{p}{q}} = \left(\cos \frac{\theta}{q} + i \sin \frac{\theta}{q} \right)^p = \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q}.$$

IX

The Measuring Rod

- 1 Plot the curve given by the equation, $y = x + 3 - 4x^2 + 2x^3$. Locate its maximum, minimum, and inflection points before plotting.
- 2 What is the angle which the line representing the equation, $y = \frac{1}{2}x + 10$, makes with the X -axis?
- 3 A stone is thrown horizontally at a speed of 100 feet per second, from a point in a boat 16 feet above the water. At what angle does the stone strike the water? ($g = 32 \text{ ft/sec}^2$)
- 4 The cable of a suspension bridge is attached to supporting pillars 200 feet apart. The lowest point of the cable is 40 feet below the points of suspension. Assuming that the cable takes the form of a parabola (a very close approximation), what is the angle between the cable and supporting pillars? (Fig. 25.)

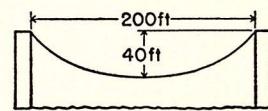


Fig. 25

MAXIMUM AND MINIMUM

- 5 For what value of x does the function, $y = x(1-x)$, have its maximum value, and what is the maximum value of the function?
- 6 What are the dimensions of the rectangle of largest area that can be cut from a triangle 10" on each side?
- 7 What is the area of the largest rectangular plate which can be cut from a piece of tin sheet in the form of an ellipse measuring 30 by 42 inches along the axes?
- 8 What are the dimensions of the heaviest right circular cylinder that a metal worker can cut from a brass cone whose altitude is 6 inches and the diameter of whose base is 4 inches?
- 9 A company wishes to produce a cylindrical tin can for use in the shipment of food to our armed forces overseas. If the can is to hold 2 quarts,

116 cubic inches, and the cost of manufacture of the can is proportional to the amount of tin used, what dimensions should be chosen so that the cost of the can will be a minimum?

- 10 A carpenter has 84 square feet of plywood to build a square box. If there is to be no top to the box, what should he make the dimensions, in order that the volume of the box should be a maximum?
- 11 What is the least amount of tin sheet needed to line a conical vat which is to hold 46 cubic feet of chemicals? What should be the dimensions of the vat?
- 12 A tinsmith has a square piece of tin with side of 40 inches. What is the volume of the largest box he can make by cutting equal squares from the corners and folding up the tin to form the sides? (Fig. 26.)
- 13 The strength of a beam varies jointly as the breadth and the square of the depth. Find the dimensions of the strongest beam that can be cut from a log whose smallest cross-section is an ellipse measuring 1 foot by 10 inches along the axes?
- 14 A locomotive uses coal at a rate which is proportional to the square of its speed and an added ton an hour. At what speed should a freight make a 500-mile trip to use as little coal as possible, if the locomotive consumes 11 tons an hour at 25 miles an hour?

- 15 The path of a projectile is given by $y = x \tan \theta - \frac{32x^2}{2V_0^2 \cos^2 \theta}$, where y is the height of the projectile, x the horizontal distance, V_0 the muzzle velocity, and θ the angle at which the gun is elevated. If $\theta = 45^\circ$, and $V_0 = 1800$ feet per second, to what height will the projectile rise?

- 16 A lake is to supply two towns, A and B , with water. The shore of the lake is a straight line, the length CD being 12 miles. A is 6 miles from the lake and B is 9 miles from the lake. If one pumping station is to be used by both of the towns, where should it be located in order that the length of the mains may be a minimum? (Fig. 27.)

- 17 A man wishes to fence off a rectangular plot of ground for a victory garden. If he allots 324 square yards to the garden, what should be the dimensions in order to use as little fencing as possible?

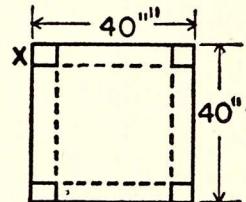


Fig. 26

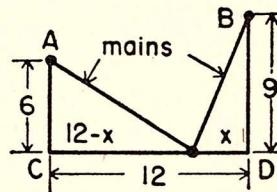


Fig. 27

DIFFERENTIALS

- 18 Find the formula for the surface area, A , of a sphere from the formula for the volume, $V = \frac{4}{3}\pi r^3$. (Note: The increase ΔV in the volume of the sphere as r is increased by Δr is, when Δr is small enough, $A \Delta r$. Hence, $A = \lim \frac{\Delta V}{\Delta r}$, etc.)
- 19 If the edge of a metal cube expands 0.25% under heat, what will be the percentage of increase of the area and the volume?
- 20 A workman measures a cylinder and finds the diameter to be 4 inches

and the height to be 7 inches. If his possible error is $\pm \frac{1}{100}$, what is the maximum error he can make in calculating the volume of the cylinder by his measurements? (Hint: Use differentials.)

21 With what percent of accuracy must the diameter of a sphere be measured so that the volume, as calculated from the measurement, shall be accurate to within $\pm 2\%$? (Hint: $V = \frac{1}{6}\pi d^3$. Take the differentials and let $dV = \pm 2\%$ of V .)

RATES

22 The distance traveled by a falling body is given in the formula, $s = \frac{1}{2}gt^2$.

Derive the formula for its speed after t seconds. (Hint: Speed is the derivative of the distance with respect to the time.)

23 The distance traveled by a particle is given by the formula, $s = 3 - 2\sqrt{t} + t^4 - 6t^3$. What are the velocity and acceleration 8 seconds after the start?

24 A circular plate expands under heat so that its radius increases 0.01 inches per minute. At what rate per minute is the area of the plate increasing when the radius is 4 inches? (Hint: $A = \pi r^2$ and $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$)

25 An enemy plane is flying at 200 miles an hour due east. Its altitude is $\frac{1}{2}$ mile and it is 1 mile due north of an anti-aircraft battery. At what rate is the plane receding from the battery one minute later? (Fig. 28.)

26 A cistern has the shape of an inverted circular cone with diameter and depth both being 8 feet. How fast is it being filled if the water is 5 feet deep and is rising at the rate of 2 inches per minute? (Hint: Find the volume in the cistern as a function of x , the depth of the water in the cistern; then differentiate this function.)

INTEGRATION

27 Find the area of the triangle formed by the X -axis, the Y -axis, and the line, $y = 2 - 2x$.

28 Find the area of the part of the parabola, $y = 2 - x^2$, which lies above the X -axis.

29 Find the area common to the parabola, $y = \frac{x^2}{2}$, and the circle, $x^2 - 4x + y^2 = 0$. (Fig. 29.)

30 A student consulted a mathematical table, and, for various values of the argument, computed the first differences (difference between adjoining values) of the function. Then he misplaced the table. He found that the first differences he had computed were all directly proportional to the arguments. What kind of table was it?

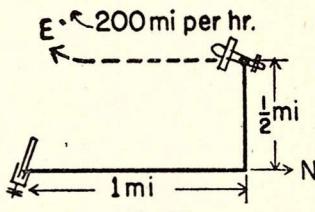


Fig. 28

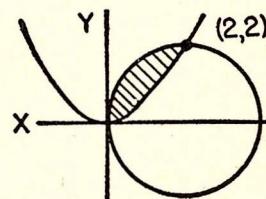


Fig. 29

31 Find the center of gravity of a triangular plate whose sides measure 3, 4, and 5 feet. (Hint: Set up the triangle with the right angle at the origin and find the center of gravity of the area bounded by the line, $3y+4x=12$, and the axes.)

32 An oval-shaped gear is composed of a parabolic section surmounted by a semi-circle with diameter of 8 inches. If the gear is 20 inches long, what will be its circumference? (Fig. 30.)

33 Develop the formula for the area of the ellipse.
(Hint: Take the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Solve for y . Integrate the area in the first quadrant and multiply by 4.)

34 A horizontal cylindrical tank with diameter of 6 ft. is half filled with an oil which weighs 55 lbs. per cu. ft. What is the force exerted by the oil on one end of the tank?

35 A type of barrage balloon can be considered an ellipsoid of revolution with axes of 20 feet, 8 feet, and 8 feet. What volume of helium is needed to inflate this balloon fully?

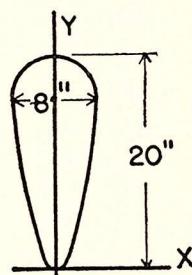


Fig. 30

Solutions to Questions and Exercises in Issue 7

ELEMENTS OF TRIGONOMETRY

ANGLES AND MEASUREMENTS

Angular Measurement

1 0.41482

2 $\theta = \frac{\text{arc}}{\text{radius}}$. $25.2\pi = \frac{40}{r}$;
 $r = \frac{4}{5\pi}$ ft. $= \frac{48}{5\pi}$ in. radius $= 3.06$ in.

12, 13 Fig. 13, superimposed upon 14, gives the triangle in Fig. 12.

18 Refer to Fig. 16.
 $\tan \alpha = -\frac{3}{4}$ $\sin \alpha = \frac{3}{5}$ $\cos \alpha = -\frac{4}{5}$
 $\cot \alpha = -\frac{4}{3}$ $\csc \alpha = \frac{5}{3}$ $\sec \alpha = -\frac{5}{4}$

19 Refer to Fig. 14.

$\tan \alpha = -\frac{\sqrt{3}}{3}$ $\sin \alpha = -\frac{1}{2}$ $\cos \alpha = \frac{\sqrt{3}}{2}$
 $\cot \alpha = -\sqrt{3}$ $\csc \alpha = -2$ $\sec \alpha = \frac{2\sqrt{3}}{3}$

25 I or II **26** II or III **27** I or III

Reducing Functions

38 $\cos \theta$ **41** $\sin \theta$ **45** $-\sin \theta$
39 $-\sin \theta$ **42** $-\cos \theta$ **46** $\cos \theta$
40 $-\cot \theta$ **43** $\cot \theta$ **47** $-\tan \theta$
44 $-\cos \theta$

54 $\sin 123^\circ = \sin (90^\circ + 33^\circ) =$
 $\cos 33^\circ = 0.83867$

$\sin 123^\circ = \sin (180^\circ - 57^\circ) =$
 $\sin 57^\circ = 0.83867$

55 $\cos 234^\circ = \cos (180^\circ + 54^\circ) =$
 $-\cos 54^\circ = -0.58779$

$\cos 234^\circ = \cos (270^\circ - 36^\circ) =$
 $-\sin 36^\circ = -0.58779$

56 $\tan 345^\circ = \tan (270^\circ + 75^\circ) =$
 $-\cot 75^\circ = -0.26795$

$\tan 345^\circ = \tan (360^\circ - 15^\circ) =$
 $-\tan 15^\circ = -0.26795$

Angular Functions

57 $45^\circ, 135^\circ$ **58** $104^\circ 30', 255^\circ 30'$
59 $116^\circ, 296^\circ$ **60** $14^\circ 30', 165^\circ 30'$
61 $63^\circ 15', 296^\circ 45'$.

PROBLEMS INVOLVING RIGHT TRIANGLES

Acute Angles

62 $\tan \alpha = \frac{AH}{EH}$ $\sin \alpha = \frac{AH}{AE}$ $\cos \alpha = \frac{EH}{AE}$
 $\cot \alpha = \frac{EH}{AH}$ $\csc \alpha = \frac{AH}{BH}$ $\sec \alpha = \frac{AE}{EH}$
 $\tan \beta = \frac{BH}{EH}$ $\sin \beta = \frac{BH}{BE}$ $\cos \beta = \frac{EH}{EB}$
 $\cot \beta = \frac{EH}{BH}$ $\csc \beta = \frac{BE}{BH}$ $\sec \beta = \frac{EB}{EH}$

Right Triangles

63 $\tan 49^\circ = \frac{t}{80}; t = 80 \times 1.5038 = 92.03$ ft.

64 Area = $\frac{1}{2}AB \cdot DC; DC = AC \sin \theta$;
 \therefore Area = $\frac{1}{2}AB \cdot AC \sin \theta$

65 $\cot 28^\circ 45' = \frac{x}{5000}; x = (5000)(1.82276);$
 $x = 9113.8$ ft.

66 $2a$ subtends a central angle of $\frac{360^\circ}{60} = 6^\circ$; $\sin 3^\circ = \frac{a}{10}; a = 0.5234$. Perimeter = $120a = 62.803$; circumference = 62.832 .

OBLIQUE TRIANGLES**Sine Law**

69 Using the law of sines, $C = 180^\circ - (49^\circ + 74^\circ) = 57^\circ$;

$$\frac{AB}{\sin 57^\circ} = \frac{4582}{\sin 74^\circ}; AB = 3998.$$

71 Let $A = 15^\circ, B = 35^\circ, C = 130^\circ$.

$$\frac{18}{\sin 15^\circ} = \frac{b}{\sin 35^\circ} = \frac{c}{\sin 130^\circ}.$$
 $\therefore b = 39.9; c = 53.3; r = 34.8$ in.
Cosine Law

72 $36+49-84 \cos \alpha = 25; \alpha = 45^\circ 35'$

73 $\angle OAC = 180^\circ - 49^\circ = 131^\circ$

$$\angle XOC = 38^\circ 09'; c = 54.8$$

74 $x_c = 43.1; y_c = 33.8; \angle XOC = 38^\circ 09'$ (ck.)

Tangent Law

75 $\frac{1}{2}(\beta+\alpha) = 78^\circ 45'. \tan \frac{1}{2}(\beta-\alpha) = \tan 78^\circ 45'$
 $\frac{7-5}{7+5} = 0.83789. \frac{1}{2}(\beta-\alpha) = 39^\circ 57'.$
 $\beta = 118^\circ 42', \alpha = 38^\circ 48'.$

Inscribed Circles

76 $s = 9; r = \frac{\sqrt{4 \cdot 3 \cdot 2}}{3} = 1.633$.

TRIGONOMETRIC IDENTITIES

77 Substitute $\cot \theta = \frac{\cos \theta}{\sin \theta}$.

78 Substitute $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

79 Carry out the multiplication:

$$\sec^2 \theta - \tan^2 \theta = 1$$

80 Square the left-hand term; then subtract $2 \sin \theta \cos \theta$ from each side.

81, 82, 83 Express in terms of sin and cos and clear of fractions

84 Apparently true when $\theta = 0^\circ, 90^\circ, 180^\circ$. However, $\sec 90^\circ$ and $\tan 90^\circ$ have no finite values. Hence, the equation holds good at only 0° and 180° .

Addition Formulas

85 $\frac{\sqrt{2}+\sqrt{6}}{4} \quad 86 \quad \frac{\sqrt{6}-\sqrt{2}}{4} \quad 87 \quad 2+\sqrt{3}$

88 $\frac{8+4\sqrt{3}}{16} + \frac{8-4\sqrt{3}}{16} = 1 \quad 89 \quad \frac{\sqrt{6}-\sqrt{2}}{4}$

90 $\cos \theta \quad 92 \cos \theta \quad 94 \sin \theta \quad 95 \quad -\cos \theta$

91 $\sin \theta \quad 93 \quad -\sin \theta \quad 95 \quad -\cos \theta$

FUNCTIONS OF MULTIPLE ANGLES**Products and Sums**

96 Use LXXI; $\alpha = 40^\circ, \beta = 20^\circ$.

97 Use LXXII.

98 Use LXXIII.

99 Use LXXVIII. 101 Use LXXX.

100 Use LXXIX. 102 Use LXXXI.

INVERSE TRIGONOMETRIC FUNCTIONS

104 $\sin \theta = 1$ when $\theta = \frac{\pi}{2}$
 $\sin \theta = 0$ when $\theta = 0$.

105 $\sin \theta = 1$ when $\theta = \frac{\pi}{2}$;
 $\sin \theta = -1$ when $\theta = -\frac{\pi}{2}$

106 $\cos \theta = 1$ when $\theta = 0$;
 $\cos \theta = -1$ when $\theta = -\pi$

107 $\tan \theta = 1$ when $\theta = \frac{\pi}{4}$;
 $\tan \theta = -1$ when $\theta = -\frac{\pi}{4}$

108 $\sin \theta = -\frac{1}{2}$ when $\theta = -\frac{\pi}{6}$
 $\cos \theta = -\frac{1}{2}$ when $\theta = \frac{2}{3}\pi$

109 $\tan (\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}) =$
 $(\frac{1}{2} + \frac{1}{3})(1 - \frac{1}{2} \cdot \frac{1}{3})^{-1} = 1 = \tan \frac{\pi}{4}$.

THE MEASURING ROD

1 Largest angle is opp. 7-rod side. Draw the alt. from that angle. $x^2 + h^2 = 9$;
 $(7-x)^2 + h^2 = 25; h = \frac{15\sqrt{3}}{14}, x = \frac{33}{14}$. Sine of side opp. the 3-rod side = $\frac{15\sqrt{3}}{14 \cdot 5} \cdot \frac{\sin A}{7} = \frac{15\sqrt{3}}{14 \cdot 5} \cdot \frac{\sin A}{7} = \frac{15\sqrt{3}}{3.70} \cdot \frac{\sin A}{2} = \frac{\sqrt{3}}{2} \cdot A = 60^\circ$ or 120° . $\triangle \neq$ equilateral; $A = 120^\circ$.

2 $\sin 143^\circ = \sin 37^\circ; e = 6.4 \cdot 0.6018 = 3.85$

3 $\cos \theta = \frac{144}{225} = 0.78750; \theta = 38.05^\circ; x = 0.66$ radians

4 Let height of mountain be x . Central angle = $\frac{110}{2880}$ radians. $A = 1.576^\circ$. $\cos A = 0.99962. x = 1.52$ mi. (approximately)

5 Draw radii to points of tangency. Drop \perp from r to R . Its length = $\sqrt{60^2 - 38^2} = 59.46$ in. Sine of $\angle = \frac{8}{60} = 0.13333$. $\angle = 7.65$. Length of belt is:
 $82.35 \times 28\pi + \frac{97.65}{360} \times 44\pi = 234.2$ By the formula, $120 + 36\pi = 233.1$ in. Error of formula, 1.1 in.

6 Divide the figure into 8 triangles.

7 The perpendicular distance from $\angle Q$ to the railroad is $40 \cos 36^\circ = 32.36$ mi.
 \therefore the part of the track =
 $2\sqrt{(34)^2 - (32.36)^2} = 20.9$ mi. will be within range.

8 Let the two segments of bridge either side of the pier = x and $208.4 - x$, and length of pier = y . $\frac{208-x}{y} = \tan 32^\circ 37'$
 $\frac{x}{y} = \tan 20^\circ 47'$; $y = 200$ ft.

9 Draw an auxiliary line the continuation of the inside of the tube. This line, $3\frac{1}{4}$ in. long, makes a rt. \triangle with a section $\frac{3}{4}$ of an inch long, of a diameter of the mouthpiece. \therefore tangent of \angle tool makes with axis of cone = $\frac{\frac{3}{4}}{3\frac{1}{4}} = 0.23077$.
 $\angle = 13^\circ$.

10 Let x = width of street. $\tan \theta = \frac{12}{x}$;
 $\tan 20 = \frac{21}{x}$; eliminating x : $4 \tan 20 - 9 \tan \theta = 0$; substituting $\frac{2 \tan \theta}{1 - \tan^2 \theta}$ for $\tan 20$: $- \tan \theta + 9 \tan 30 = 0$; $\tan \theta = \frac{1}{3}$; Going back to $\tan \theta = \frac{12}{x}$; $x = 36$ ft.

11 \angle between the two lights as seen from the ship is $22^\circ 40'$. Use law of cosines. $x^2 = (28.4)^2 + (16.3)^2 - 2(28.4)(16.3) (\cos 22^\circ 40')$. $x = 14.7$ mi.

12 $\text{arc tan } \frac{1}{2}$, $\tan^{-1} \frac{1}{2}$

13 Call the width of the stream x and let the 200' length be divided into y and $200-y$ by the point opposite the tree. $\frac{x}{y} = \tan 79^\circ 42'$; $\frac{x}{200-y} = \tan 23^\circ 12'$. Solving these equations, $x = 79.5$ ft.

14 Apply the law of sines. Let the distance be x , θ the angle ladder makes with ground, and φ the angle ladder makes with house. $\frac{\sin(90^\circ + 12^\circ 39')}{32.75} = \frac{\sin \theta}{25}$; $\frac{\sin \theta}{x} = \frac{\sin \varphi}{25}$; $\theta = 48^\circ 9'$; $\varphi = 90^\circ - 48^\circ 9' - 12^\circ 39' = 29^\circ 12'$. By logs, $x = 16.37'$.

15 Tank is 4 yd. long. $R = \frac{4 \times 1000}{2} = 2000$ yd.

16 Distances to ship from base of lighthouse are: $A = 200 \tan 79^\circ 46' = 1107.9'$; $B = 200 \tan 78^\circ 50' = 1013.2$. Apply law of cosines, letting x be required distance. $x^2 = (1107.9)^2 + (1013.2)^2 - 2(1107.9)(1013.2) \cos 127^\circ 14'$. $x = 1901'$.

17 Call distance from foot of rock to ship, y , and height of rock, x . $\frac{y}{x} = \tan 79^\circ 15'$; $\frac{y}{85+x} = \tan 53^\circ 45'$. Solv-

ing, we get, $x = \frac{85 \tan 53^\circ 45'}{\tan 79^\circ 15' - \tan 53^\circ 45'}; x = 29.7'$.

18 36.74 ft. (See prob. 10 for solution.)

19 Let x be distance from corner. $\tan 51^\circ 30' = \frac{x}{3}; x = 3.77$ ft.

20 $r = h \tan 35^\circ$; $V = \frac{\pi}{3} \tan^2 35^\circ h^3 = 18$. By logs, $h = 3.273$ ft.

21 Let x be one half width at top. $x = 4 \tan 63^\circ$; $4x = 16 \tan 63^\circ$. (the cross-section area, A). $\frac{V}{A} = \text{the length} = \frac{487}{16 \tan 63^\circ} = (\text{by logs}) 15.51$ ft.

22 Batteries are 1140 and 1520 yd. from observer. Applying law of cosines, letting x be distance between batteries, we get $x^2 = (1140)^2 + (1520)^2 - 2(1140)(1520) \cos 42^\circ$. $x = 1017$ yd.

23 $\angle BDC = 43^\circ 3'$. By law of sines, $CD = \frac{\sin 62^\circ 46' \cdot 792.6}{\sin 43^\circ 3'} = (\text{by logs}) 1032.4$ ft.

24 By law of cosines, $GT^2 = (13820)^2 + (8405)^2 - 2(3820)(8405) \cos 133^\circ$. $GT = 11390$ yd. $\angle PGT = (\text{law of sines}) 32^\circ 8'$

25 120 mph = 176 ft. per sec.
 $h = \frac{120 \cdot 176}{\cot 20^\circ - \cot 43^\circ} = 12608$ ft.

26 40 ft. (For solution see prob. 14.)

27 $y = O$; $x = 12 \times 5280$ ft. $32x^2 = \frac{2V_0^2 \sin \theta \cos \theta x}{\sin 2\theta} = \frac{32 \cdot 12 \cdot 5280}{1600 \cdot 1600} = .792$; $\theta = 26^\circ 11'$

28 Use $A = \frac{ab}{2} \sin C = 1109$ sq. rd.

29 $KE = \frac{I\omega^2}{2}$; $\omega = \frac{63}{60} \times 2\pi$ radians per sec.
 $KE = \frac{7(2.1\pi)^2}{2} = 152.34$ (by logs)

30 $\sqrt{18^2 + 14^2} - 2 \cdot 18 \cdot 14 \cos 100^\circ = 24.6$ kn.

31 Let angle be 20. $\sin \theta = \frac{4.615}{7} = 0.65929$;
 $\theta = 41^\circ 15'$; $20 = 82^\circ 30'$.

32 $h = 19 \cot 64^\circ = 9.267$ mi.

33 Method same as in prob. 23. Undoubtedly more than two auxiliary lines would have to be used to circle a mountain.

34 Assume chord = arc = $\frac{40\pi}{120} = 1.047$ mi.

35 Length of each tooth is $2x$. $x = 0.2 \tan 29^\circ 30'$; $2x = 0.226308$ in. $\frac{19}{0.226308} = 83+$, or 83 teeth.

36 Angle is $\cot^{-1} 0.75 = 53^\circ 08'$.

Odd Problems for Off Hours

Up the Shaft

52 A group of prisoners were shut up in a room in the center of whose ceiling was a circular opening leading into a shaft. For want of better amusement, they proceeded to play with long poles which were lying on the floor of the room. What was the length of the longest straight stick which they could put up the shaft? (This may be solved algebraically without knowing the dimensions of the room or of the shaft. To give concreteness to the problem, let us take 18 feet as the height of the room and 10 feet as the diameter of the shaft.)

It's a Long Road

53 Desiring to take the enemy by surprise, the leader of a scouting party decides to march his group up a winding path leading to the top of a cone-shaped hill 200 feet high. If the base of the hill is 100 feet in diameter and

The Solutions to These Puzzles Will Appear in Issue Number Ten

ANSWERS TO PUZZLE-PROBLEMS IN ISSUE NUMBER SIX

46 The answers may be attained by actual measurement or by laying a straight edge along the lines. The irregular objects in Fig. 67 are equal in size. One's eye has a tendency to compare the short curve at the right of the first object with the long curve at the left of the second object, thereby confusing the points of similarity in the two objects.

47 The solution of this involves a geodesic (Issue Number Six, page 357). By drawing three rectangles, each 30×12 , one below the other, we get a "flattening out" of ceiling, wall, and floor. A square 12×12 at the left of the top rectangle and a similar square at the right of the bottom rectangle enable us to locate the positions of the spider and the fly. From these positions, draw lines parallel to sides of rectangles, obtaining 24' and

the road slopes at 5 per cent, how far will the scouts have to travel in walking along the road?

Crossed Ladders

54 Firemen engaged in rescuing persons from two burning buildings across a narrow alley from each other placed ladders, braced against the wall of the opposite building to points 10 and 14 feet, respectively, above the pavement. How far above the pavement was the point where the ladders crossed each other?

A Dozen Matches

55 Place 12 match sticks as shown in the diagram (Fig. 31). By moving

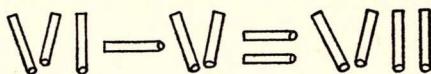


Fig. 31

one match, make the equation true.

32' for the legs of our right triangle. These have a 3:4 relationship, since each is a multiple of 8. The hypotenuse would be the fifth multiple of 8, or 40'.

48 Take *E* as mid-point of *AB*, *F* as mid-point of *BD*, *G* as mid-point of *CD*, and *H* as mid-point of *AC*. Draw *AG*, *BH*, *CF*, and *DE*. Cut along these lines. You now have one perfect square, four trapezoids, and four triangles. Fitting a triangle and a trapezoid together will give you another square; thus, when all are matched, you have five squares whose total area equals the area of the given square.

50 By placing the rifle diagonally in a rectangular package, the sportsman reduces the length of his package sufficiently to bring it within the requirements. (An adaptation of the Pythagorean theorem.)

Strange Ways With Numbers

We call upon trigonometry to assist us in plotting the curves of several loci which occur in connection with rotating circular objects, such as wheels and gears.

THE TROCHOID

31 Let us imagine a straight shaft attached at the center of a circle, such as the line, CD , in Fig. 32.

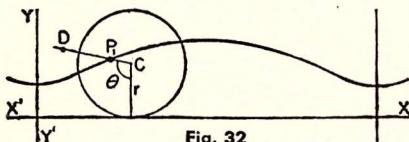


Fig. 32

On this line, we take the distance, CP_1 , less than the length of the radius r . We shall designate the angle between the radius (drawn perpendicularly to the base on which the circle rolls) and the line, CD , as θ , the measurement being in radians. As we rotate the circle, the locus of the point, P_1 , is an undulating line, known as a *curtate cycloid*, which never touches the base line.

Let us take, also, the distance, CP_2 , greater than r . The point, P_2 , will

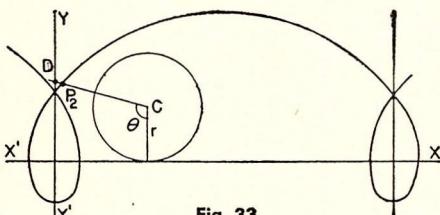


Fig. 33

trace a curve which has a loop between every two arches, and will have *nodes* (a double point of intersection) at $\theta=0$ and at every point where θ is an even multiple of π . This curve is known as a *prolate cycloid*.

The parametric equations for trochoids are:

$$x = r - p \sin \theta$$

$$y = r - p \cos \theta$$

If we take on the shaft a distance, CP_3 , equal to r , we get a special case, known simply as the *cycloid*. This makes a series of arches, at intervals of $2\pi r$, with a cusp at every point where the cycloid touches the base line.

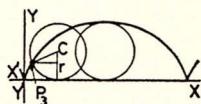


Fig. 34

The parametric equations for the cycloid are (since r and p are equal):

$$x = r(\theta - \sin \theta)$$

$$y = r(1 - \cos \theta)$$

THE EPICYCLOID

32 If we imagine the circle on which our point, P , is located as rolling around the circumference of another circle, we have an interesting figure. Designating the radius of the fixed circle as r_1 and the radius of the revolving circle as r_2 , we may readily adapt the parametric equations to fit this special case:

$$x = (r_1 + r_2) \cos \theta - a \cos \frac{(r_1 + r_2)\theta}{a}$$

$$y = (r_1 + r_2) \sin \theta - a \sin \frac{(r_1 + r_2)\theta}{a}$$

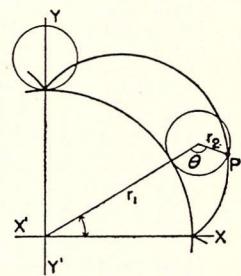


Fig. 35

When $r_1 = r_2$, the curve has one arch; when $r_1 = \frac{r_2}{2}$, the curve has two arches;

when $r_1 = \frac{r_2}{n}$, the curve has n arches.

There is a *cusp* at every point where the curve touches the fixed circle.

CASSINIAN OVALS

33 Taking its name from the mathematician, Cassini, this figure denotes the locus of the vertex of a triangle when the product of the sides adjacent to the vertex is a constant and the length of the opposite side is fixed. Letting k represent the constant and $2a$ the length of the fixed side, we form the equation,

$$[(x+a)^2+y^2][(x-a)^2+y^2]=k^4.$$

If $k^2 > a^2$, the curve consists of one oval, as shown in Fig. 36. If $k^2 < a^2$, we get two distinct ovals. When

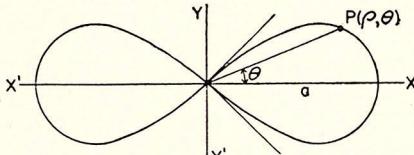


Fig. 37

$k^2 = a^2$, we get two symmetrical nodes, known as the *lemniscate*.

Bernouilli, who first studied the properties of the lemniscate, developed for it the equation,

$$\rho^2 = a^2 \cos 2\theta.$$

In Cartesian terminology, we may express it as:

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

THE LIMAON

34 Pascal's limacon is the name given to the locus of a point on a line, at a fixed distance from the intersection of the line with a fixed circle as the line revolves about a point on the circle. Taking the diameter through the fixed circle as the polar axis, and representing its length by d , the fixed

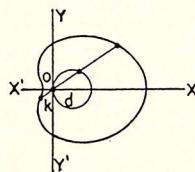


Fig. 38

point as the pole, O , and the fixed distance as k , we get as the equation of the limacon,

$$r = d - \frac{k^2}{2} \cos \theta.$$

When $d > k$, the curve has two loops, as in Fig. 39. When $d < k$, there is one loop, as in Fig. 38. As k increases, the curve tends toward the shape of a circle.

If $k = 2d$, we get a special case of the limacon, which is a one-looped epicycloid, whose equation is

$$r = \frac{2d \sin \varphi}{2}$$

$$r = d(1 - \cos \varphi).$$

This, from its heart-shaped appearance, is known as the *cardioid*.

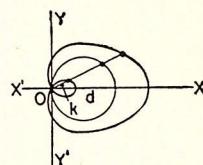


Fig. 39

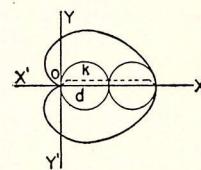


Fig. 40

THE STROPHOID

Another interesting figure is developed when a line passing through a fixed point is moved so that the distance from the describing point to the intersection of the line with the X -axis is equal to the Y -intercept. Taking the coordinates of the fixed point, A , as $(-a, 0)$, we determine the equation of the *strophoid* to be:

$$y^2 = \frac{x^2(x+a)}{x-a}.$$

Note that $P_2E = EP_1 = OB$. The asymptote of the curve is designated by the dotted line, MM' .

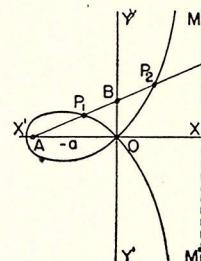


Fig. 41

Tables and Formulas

**TABLE XLI
STANDARD FORMS OF DERIVATIVES**

$\frac{dx^n}{dn} = nx^{n-1}$	$\frac{d(\tan x)}{dx} = \sec^2 x$
$\frac{d(\log_e x)}{dx} = \frac{1}{x}$	$\frac{d(\cot x)}{dx} = -\csc^2 x$
$\frac{de^x}{dx} = e^x$	$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$
$\frac{da^x}{dx} = a^x \log_e a$	$\frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d(\sin x)}{dx} = \cos x$	$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$
$\frac{d(\cos x)}{dx} = -\sin x$	$\frac{d(\cot^{-1} x)}{dx} = -\frac{1}{1+x^2}$

If u and v are any functions of x , and c is a constant,

$$\begin{aligned}\frac{d(u+v)}{dx} &= \frac{du}{dx} + \frac{dv}{dx} & \frac{d(u/v)}{dx} &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\ \frac{d(uv)}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} & \frac{d}{dx} f(u) &= \frac{d}{du} f(u) \frac{du}{dx}\end{aligned}$$

**TABLE XLII
INTEGRALS OF BASIC FUNCTIONS***

- I $\int adx = ax$
- II $\int \frac{dx}{x} = \log_e x$
- III $\int x^m dx = \frac{x^{m+1}}{m+1}$, when m is not -1
- IV $\int e^x dx = e^x$
- V $\int a^x \log_e a dx = a^x$
- VI $\int \log_e x dx = x \log_e x - x$
- VII $\int \frac{dx}{1+x^2} = \tan^{-1} x$

- VIII $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$
- IX $\int \sin x dx = -\cos x$
- X $\int \cos x dx = \sin x$
- XI $\int \tan x dx = -\log_e \cos x$
- XII $\int \cot x dx = \log_e \sin x$
- XIII $\int \sec x dx = \log_e \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$
- XIV $\int \csc x dx = \log_e \tan \frac{1}{2} x$

In the following, u and v represent any functions of x , and c is a constant.

$$\begin{aligned}XV \int (u+v) dx &= \int u dx + \int v dx \\ XVI \int c u dx &= c \int u dx \\ XVII \int u dv &= uv - \int v du\end{aligned}$$

$$XVIII \int f(u) dx = \int \frac{f(u)}{du} du$$

* A Short Table of Integrals, by B. O. Pierce (Ginn and Co.), contains several hundred frequently encountered forms.

Size 6 $\frac{1}{4}$ x 9 $\frac{1}{2}$ Inches. Binders open flat, and the patent fasteners permit any issue to be inserted (or removed) at any time without disturbing the others.



2 of these Binders
Will Hold Your
Entire Series of
**PRACTICAL
MATHEMATICS**

(See Other
Picture on
Outside
of this
Cover)

TOO GOOD A BARGAIN TO LAST!

As a special service to members, we had a prominent manufacturer especially create this De Luxe Artcraft Binder, to be sold at the extremely low price of 2 for only \$1.00. Unfortunately, wartime shortages limit production of these Binders, and we are informed that the only deliveries that can be guaranteed are on orders that are placed AT ONCE, while the manufacturer still has materials. Prompt action on your part is therefore necessary.

our offer is STILL

**2 for \$1 (a \$3
only value)**

**while the limited
supply is available!**

**MAIL THE COUPON ATTACHED OR,
ORDER ON YOUR OWN STATIONERY**

No matter how fast and how thoroughly you absorb the instructions in each issue of **PRACTICAL MATHEMATICS**, you will certainly want to safeguard from loss or damage every single copy. Even years from now, regardless of how far you progress in Mathematics, you will have frequent occasion to refer to this veritable Encyclopedic Storehouse of Mathematics Instruction.

By filing the entire series in these De Luxe Artcraft Binders, you may convert your loose issues into a priceless, up-to-the-minute Ready Reference File of Practical Mathematics—to help you to quick, easy, accurate solution of any mathematical problem that may arise in your work in any field.

NATIONAL EDUCATIONAL ALLIANCE, INC.
37 West 47th Street, New York, N.Y.

I enclose \$1. Please send me 2 **PRACTICAL MATHEMATICS** De Luxe Morocco-Grained Artcraft Binders, which I understand are all I need to hold my entire series of **PRACTICAL MATHEMATICS** Lecture-Groups.

Name

Address

City and State

Warning! Time Is Short

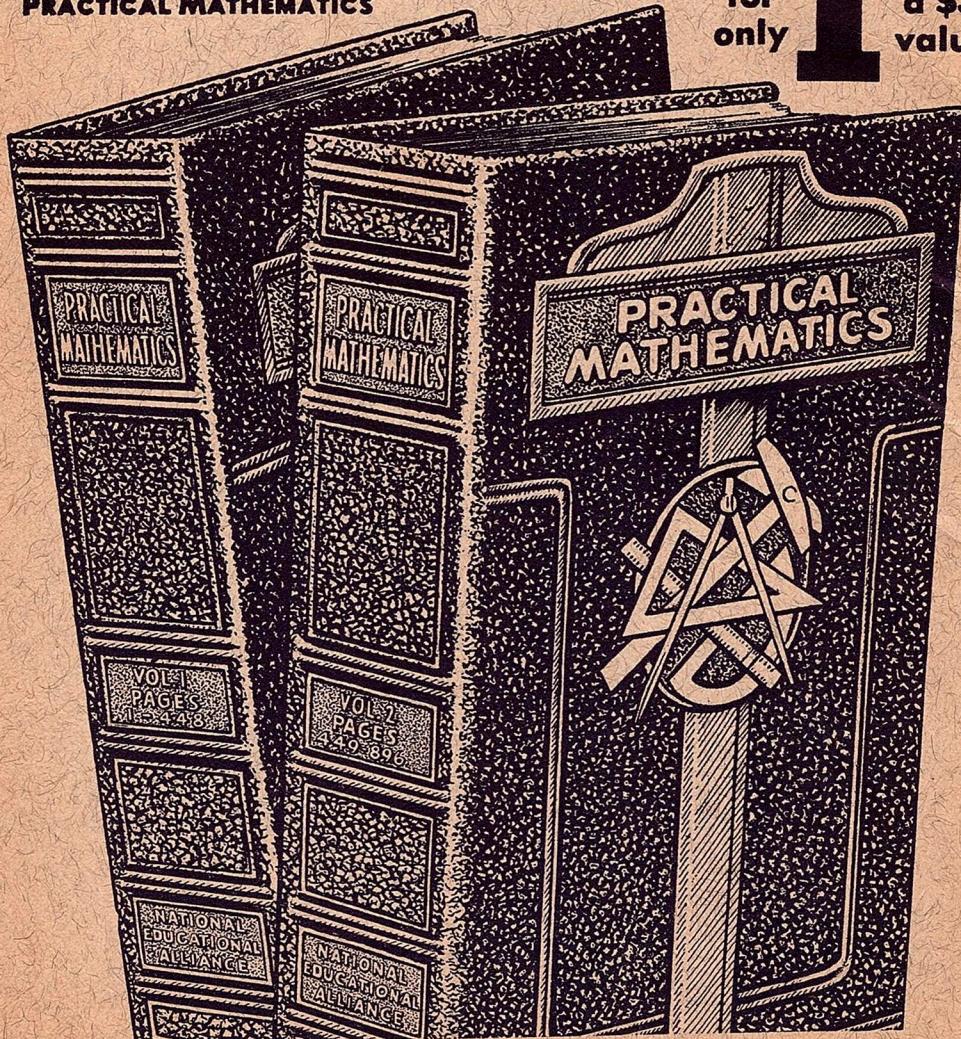
Our Member's Service Department repeats this warning sent along by the manufacturer of these binders: Wartime shortages have severely limited production. The only binders that can positively be delivered at this extremely low price are those which are ordered at once, while the manufacturer still has materials!

These 2 De Luxe Artcraft Binders
Will Hold Your Complete File of
PRACTICAL MATHEMATICS

• both

\$1
for
only

a \$3
value



SEE INSIDE OF THIS COVER FOR HANDY ORDER FORM!